UNIT - I

Electrostatics:

1. Vector Algebra
2. Coordinate System
3. Vector Calculus
4. Coulombs law
5. Electric field Intensity (E)
6. Electric Field due to Continuous charge Distribution
   - → Due to Point charge
   - → Due to line charge
   - → Due to Surface charge
   - → Due to Volume charge
7. Electric flux & Electric flux Density (D)
8. Relation between E and D
9. Gauss's law and its Limitation
10. Applications of Gauss's law
11. Electric Potential (V)
12. Relation between E and V
13. Maxwell's Two equations for Electrostatic fields
14. Poisson's and Laplace's equations
15. Dipole and Dipole Moment
16. Electrostatic Energy and Energy Density
17. Capacitance
    - → Parallel plate capacitor
    - → coaxial (or) cylindrical capacitor
    - → Spherical capacitor

* YVNR *

EMF-1

ELECTRO MAGNETIC FIELDS & WAVES

Electro Magnetic Field :

It is a branch of physics in which electric & magnetic fields are studied.

→ Electromagnetics (EM) may be regarded as the study of the interactions between electric charges at rest and in motion.

→ EM principles find applications in various disciplines such as Antennas, Microwaves, satellite communications, fiber optics, RADARS, Radio, Telivision, Telephone .. etc.

* Vector Analysis simplifies the formulation of various laws of EM fields. Therefore the basic concepts of vector algebra, coordinate system and vector calculus should be studied before entering into the EM fields.

1. Vector Algebra :

It is a mathematical tool that can be used to express the quantities in different coordinate systems with magnitude and direction.

→ A quantity can be either a scalar (or) a vector.

Scalar : It is a quantity that has only magnitude.

Ex Time, Temperature, mass, distance, electric potential. etc.,

A scalar is represented simply by a letter like A, B, U & V.

Vector: It is a quantity that has both magnitude & direction.

Ex Velocity, Force, displacement and electric field intensity.

A vector is represented by a Letter with an arrow on top of it, Such as $\vec{A}$ and $\vec{B}$ (or) bold Letters such as **A** and **B**.

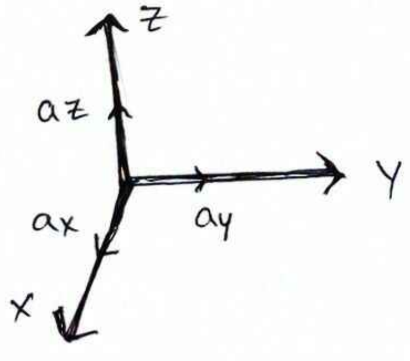
Field: Any quantity may have either scalar (or) vector Field.

A field is a function that specifies a particular quantity everywhere in a region.

Examples of scalar fields are
1. Temperature distribution in a building
2. Sound intensity in a theater
3. Electric Potential in a region

Examples of vector fields are
1. Fan rotation
2. current flow
3. Electric forces
4. Magnetic forces.

Unit Vector:



A unit vector $\vec{a}_A$ along $\vec{A}$ is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along $\vec{A}$. That is $\boxed{\vec{a}_A = \frac{\vec{A}}{|A|} = \frac{\vec{A}}{A}}$.

* $\vec{A}$ is a vector and it is represented in cartesion coordinates as (A_x, A_y, A_z) (or) $A_x a_x + A_y a_y + A_z a_z$

→ where A_x, A_y and A_z are called components (or) magnetudes of A in the x, y & z directions respectively

→ a_x, a_y and a_z are called unit vectors in the x, y and z-directions respectively.

The magnitude of $\vec{A}$ is given by

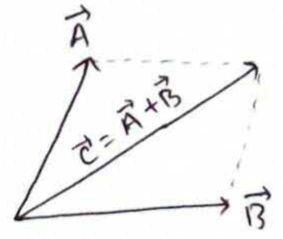
$$\boxed{|A| = \sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$\therefore$ The unit vector along $\vec{A}$ is given by

$$\boxed{\vec{a}_A = \frac{A_x a_x + A_y a_y + A_z a_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}}$$

Here $\boxed{|\vec{a}_A| = 1}$; Hence $\boxed{\vec{A} = |A|\, \vec{a}_A}$

Vector Addition and Subtraction :



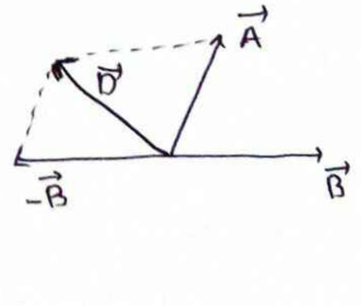
Two vectors $\bar{A}$ & $\bar{B}$ canbe added together to give another vector C ;

i.e., $\boxed{\vec{C} = \vec{A} + \vec{B}}$

→ Vector addition is carried out component by component. Thus, if $\vec{A} = (A_x, A_y, A_z)$ & $\vec{B} = (B_x, B_y, B_z)$ then

$$\vec{C} = \vec{A} + \vec{B} = (A_x + B_x) a_x + (A_y + B_y) a_y + (A_z + B_z) a_z.$$

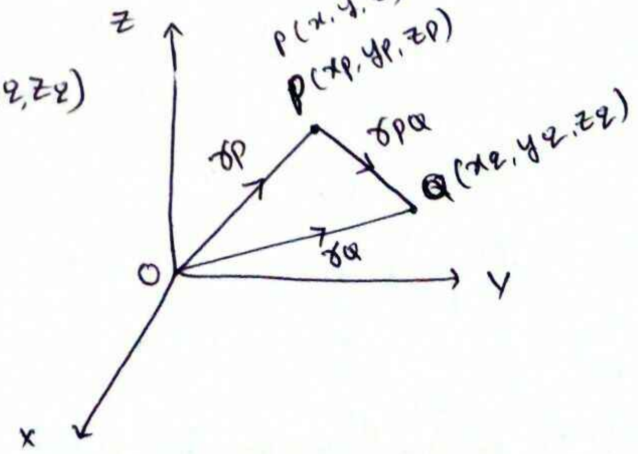
→ Vector Subtraction is similarly carried out as



$$\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

$$\vec{D} = (A_x - B_x) a_x + (A_y - B_y) a_y + (A_z - B_z) a_z.$$

Position and Distance Vectors :



consider two points $P(x_P, y_P, z_P)$ & $Q(x_Q, y_Q, z_Q)$

$$r_P + r_{PQ} - r_Q = 0$$

$$r_{PQ} = r_Q - r_P$$

→ Position vector (or) radius vector is a vector which represents the distance from the origin to the point p.

$$\boxed{\therefore \vec{A} = \vec{OP} = x a_x + y a_y + z a_z.}$$

∴ Here OP is a Position vector.

→ Distance vector is a vector which represents the distance from one point to another point.

$$\boxed{\vec{r}_{PQ} = \vec{r}_Q - \vec{r}_P}$$

∴ Here r_{PQ} is a distance vector.

$$\boxed{r_{PQ} = (x_Q - x_P) a_x + (y_Q - y_P) a_y + (z_Q - z_P) a_z}$$

Vector Multiplication:

→ When two vectors A & B are multiplied, the result is either a scalar (or) a vector depending on how they are multiplied.

→ There are two types of vector multiplication.
1. Scalar product (A·B)
2. Vector product (A×B)

1. Scalar product (or) Dot product: of two vectors is defined as the product of the magnitudes of A and B and cosine of the angle between them.

$$\boxed{\vec{A}\cdot\vec{B} = |A|\,|B| \cos\theta_{AB}} \quad \& \quad \boxed{\theta_{AB} = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{|A|\,|B|}\right)}$$

→ Dot product is a scalar quantity.

If $\vec{A} = (A_x, A_y, A_z)$ & $\vec{B} = (B_x, B_y, B_z)$ then $\boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$

Properties:
i) $A\cdot B = B\cdot A$ Commutative law

ii) $A\cdot(B+C) = A\cdot B + A\cdot C$ Distributive law

$A\cdot A = |A|^2 = A^2$

iii) $a_x\cdot a_x = a_y\cdot a_y = a_z\cdot a_z = 1$

$a_x\cdot a_y = a_y\cdot a_z = a_z\cdot a_x = 0$

2. Vector product (or) Cross product

The cross product of two vectors $\vec{A}$ & $\vec{B}$ is defined as the magnitudes of A & B and Sin of the angle between them.

$$\boxed{\vec{A} \times \vec{B} = |A|\,|B| \sin\theta_{AB}\, a_n}$$

$a_n \rightarrow$ unit vector normal (Perpendicular) to the plane containing A & B.

Let $A = (A_x, A_y, A_z)$ (or) $A_x a_x + A_y a_y + A_z a_z$

$B = (B_x, B_y, B_z)$ (or) $B_x a_x + B_y a_y + B_z a_z$, then

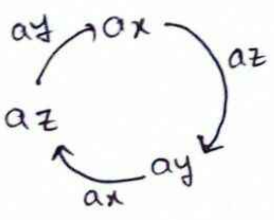
$$\boxed{A \times B = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}$$

Properties:

i) $A \times B = -B \times A$ anti commutative

ii) $A \times (B + C) = A \times B + A \times C$ distributive.

$A \times A = 0$

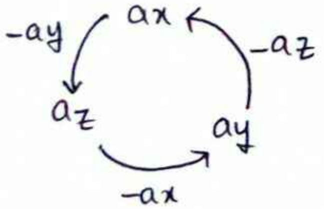


clock wise moving leads to positive

$a_x \times a_y = a_z$

$a_y \times a_z = a_x$

$a_z \times a_x = a_y$



anti clock wise moving Leads to negative

$a_x \times a_y = -a_z$

$a_y \times a_z = -a_x$

$a_z \times a_x = -a_y$

$a_x \times a_x = a_y \times a_y = a_z \times a_z = 0.$

Scalar Triple Product :

→ If three vectors $\vec{A}, \vec{B}$ & $\vec{C}$ are given then the scalar triple product is defined as

$$\boxed{A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B)}$$

If $A = (A_x, A_y, A_z)$, $B = (B_x, B_y, B_z)$ & $C = (C_x, C_y, C_z)$ then

$$A\cdot(B\times C) = [A B C] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$B\cdot(C\times A) = [B C A]$$

$$C\cdot(A\times B) = [C A B]$$

→ Scalar triple product is a scalar quantity.

Vector Triple product :

→ If three vectors $\vec{A}, \vec{B}$ & $\vec{C}$ are given then the vector triple product is defined as

$$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$

$$B\times(C\times A) = C(B\cdot A) - A(B\cdot C)$$

$$C\times(A\times B) = A(C\cdot B) - B(C\cdot A) \quad \text{and also}$$

$$(A\cdot B)C = C(A\cdot B)$$

→ Vector triple product is a vector quantity.

Components of a Vector :

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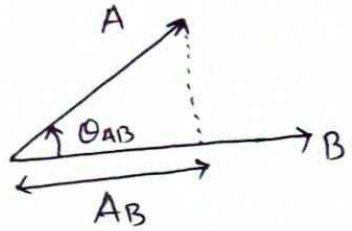
A direct application of vector product is its use in determing the projection (or component) of a vector in a given direction. The projection can be scalar (or) vector.



Given a vector $\vec{A}$, we define the scalar component A_B of $\vec{A}$ along vector $\vec{B}$ as

$$A_B = \vec{A}\cos\theta_{AB} = |A||a_B|\cos\theta_{AB}$$

* $$\boxed{A_B = \vec{A}\cdot\vec{a}_B}$$

The vector component $\vec{A}_B$ of $\vec{A}$ along $\vec{B}$ is simply the scalar-component in above equation multiplied by a unit vector along $\vec{B}$: i.e.,

* $$\boxed{\vec{A}_B = A_B\,\vec{a}_B = (\vec{A}\cdot\vec{a}_B)\,\vec{a}_B}$$

Problem

Given vectors $A = -a_x + 2a_y + 2a_z$, $B = 2a_x - a_y + 2a_z$ and $C = 2a_x + 2a_y - a_z$. find a) $|A|, |B|$ & $|C|$ b) $A\cdot B$, $B\cdot C$ & $C\cdot A$

c) $A\times B$, $B\times C$ & $C\times A$ d) θ_{AB}, θ_{BC} & θ_{CA}

e) $A\cdot(B\times C)$, $B\cdot(C\times A)$ & $C\cdot(A\times B)$ and f) $A\times(B\times C)$.

Sol a) $|A| = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1+4+4} = \sqrt{9} = 3$

$|B| = \sqrt{4+1+4} = 3$

$|C| = \sqrt{4+4+1} = 3$

b) $A\cdot B = A_xB_x + A_yB_y + A_zB_z = (-1)(2) + 2\cdot(-1) + 2(2) = -2-2+4 = 0$

$B\cdot C = 4-2-2 = 0$

$C\cdot A = -2+4-2 = 0$

c) $$A\times B = \begin{vmatrix} a_x & a_y & a_z \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= a_x(4+2) - a_y(-2-4) + a_z(+1-4)$$

$$= 6a_x + 6a_y - 3a_z$$

$$B\times C = \begin{vmatrix} a_x & a_y & a_z \\ 2 & -1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = a_x(1-4) - a_y(-2-4) + a_z(4+2)$$
$$= -3a_x + 6a_y + 6a_z$$

$$C\times A = \begin{vmatrix} a_x & a_y & a_z \\ 2 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = a_x(4+2) - a_y(4-1) + a_z(4+2)$$
$$= 6a_x - 3a_y + 6a_z.$$

d). $$\theta_{AB} = \cos^{-1}\left(\frac{A\cdot B}{|A||B|}\right)$$
$$= \cos^{-1}\left(\frac{0}{3\times 3}\right)$$
$$= 90^\circ$$

$$\therefore \theta_{AB} = \theta_{BC} = \theta_{CA} = 90^\circ.$$

e) $$A\cdot(B\times C) = [ABC] = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & 1 \end{vmatrix}$$
$$= -1(1-4) - 2(-2-4) + 2(4+2)$$
$$= 3 + 12 + 12$$
$$= 27$$

$\Rightarrow A\cdot(B\times C)$ (or)

$$\Rightarrow A\cdot(-3a_x + 6a_y + 6a_z) = (-a_x + 2a_y + 2a_z)\cdot(-3a_x + 6a_y + 6a_z)$$
$$= 3 + 12 + 12$$
$$= 27.$$

$$\therefore A\cdot(B\times C) = B\cdot(C\times A) = C\cdot(A\times B) = 27.$$

f). $$A\times(B\times C) = B(A\cdot C) - C(A\cdot B)$$
$$= B(0) - C(0)$$
$$= 0 - 0$$
$$= 0.$$

2. COORDINATE SYSTEMS :

Coordinate System is defined as a system which is used to represent a Point in Space. These are basically three types.

1. Cartesian coordinate System
2. Cylindrical coordinate System
3. Spherical coordinate System.

→ A Point (or) Vector can be represented in any coordinate System, which may be orthogonal.

→ An orthogonal System is one in which the coordinates are mutually Perpendicular.

1. Cartesian (or) Rectangular coordinate System :

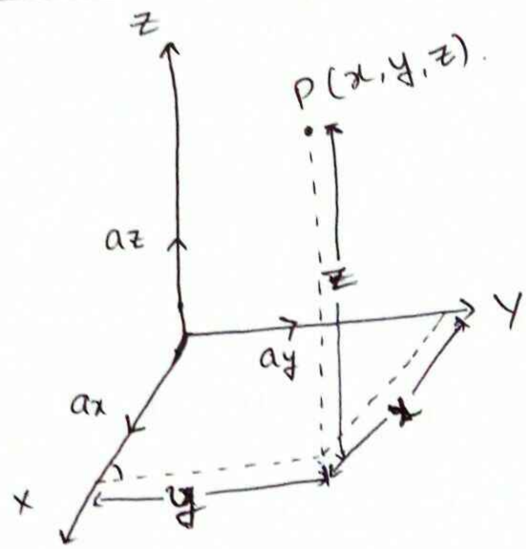
A Point P can be represented as (x, y, z) shown in figure.

→ The range of coordinate variables x, y and z are

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < z < \infty$$



→ A vector $\vec{A}$ in cartesian coordinates can be written as

$$(A_x, A_y, A_z) \quad \text{(or)} \quad A_x a_x + A_y a_y + A_z a_z$$

where a_x, a_y and a_z are unit vectors along the x, y and z directions respectively as shown in figure. In $P(x, y, z)$

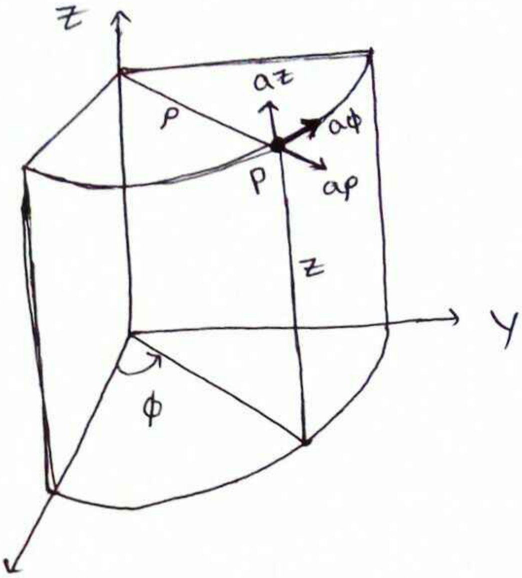
x, y, z are distance from origin to point P along x, y, z axises respectively.

x → distance of Point P along the x-axis from the origin.

y → " " " y-axis " "

z → " " " z-axis " "

2. Cylindrical (or) Circular Cylindrical Coordinates : (ρ, ϕ, z)



A Point "P" can be represented as (ρ, ϕ, z) shown in figure.

- In figure ρ is the radius of the cylinder passing through P (or) the radial distance from the z-axis.
- ϕ, called the azimuthal angle is measured from the x-axis in the x-y plane.
- z is the same as in the Cartesian coordinate system.

The range of coordinate variables (ρ, ϕ, z) are

$$0 \le \rho < \infty$$
$$0 \le \phi < 2\pi$$
$$-\infty < z < \infty$$

→ A vector $\vec{A}$ in cylindrical coordinates can be written as

$$(A_\rho, A_\phi, A_z) \quad \text{(or)} \quad A_\rho a_\rho + A_\phi a_\phi + A_z a_z$$

where a_ρ, a_ϕ and a_z are unit vectors in the ρ, ϕ and z directions shown in figure.

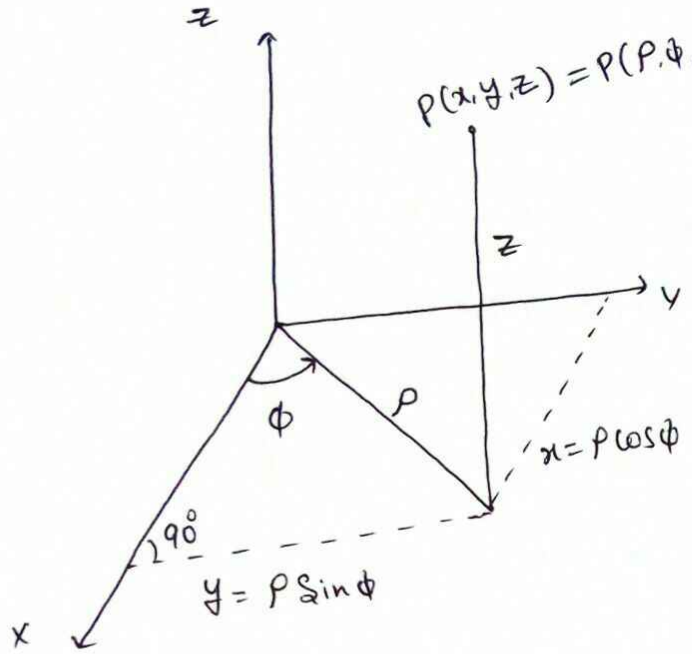
→ The magnitude of $\vec{A}$ is $\quad |A| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$

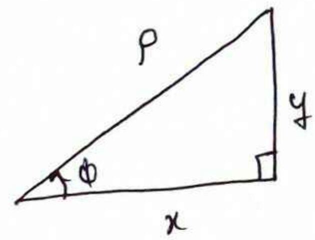
Properties

$$a_\rho \cdot a_\rho = a_\phi \cdot a_\phi = a_z \cdot a_z = 1$$
$$a_\rho \cdot a_\phi = a_\phi \cdot a_z = a_z \cdot a_\rho = 0$$

$$a_\rho \times a_\phi = a_z$$
$$a_\phi \times a_z = a_\rho$$
$$a_z \times a_\rho = a_\phi$$

Relation between the variables (x, y, z) & (ρ, ϕ, z):





$\sin\phi = \frac{y}{\rho} \Rightarrow \boxed{y = \rho \sin\phi}$

$\cos\phi = \frac{x}{\rho} \Rightarrow \boxed{x = \rho \cos\phi}$

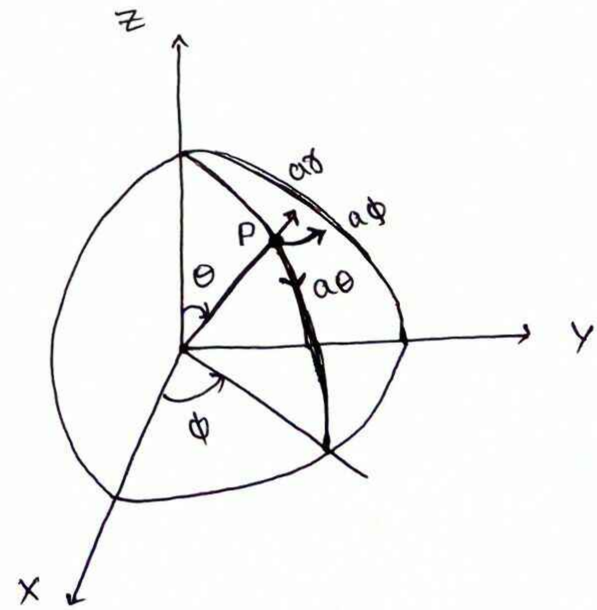
$\boxed{z = z}$

$\rho^2 = x^2 + y^2 \Rightarrow \boxed{\rho = \sqrt{x^2 + y^2}}$

$\tan\phi = \frac{y}{x} \Rightarrow \boxed{\phi = \tan^{-1}\frac{y}{x}}$

$\boxed{z = z}$

3. Spherical Coordinates: (r, θ, ϕ)



A Point P can be represented as (r, θ, ϕ) shown in figure

⇒ where r is the distance from the origin to Point P (or) radius of sphere centered at the origin and passing through P.

⇒ θ called elivation angle is the angle between the z-axis and the position vector of P.

⇒ ϕ is same as in the cylindrical coordinate system.

→ The range of coordinate variables (r, θ, ϕ) are

$$0 \leq r < \infty$$
$$0 \leq \theta < \pi$$
$$0 \leq \phi < 2\pi$$

→ A vector $\vec{A}$ in spherical coordinates can be written as

$$(A_r, A_\theta, A_\phi) \quad \text{(or)} \quad A_r a_r + A_\theta a_\theta + A_\phi a_\phi$$

where A_r, A_θ and A_ϕ are the unit vectors in r, θ and ϕ directions shown in figure.

→ The magnitude of $\vec{A}$ is $|A| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$

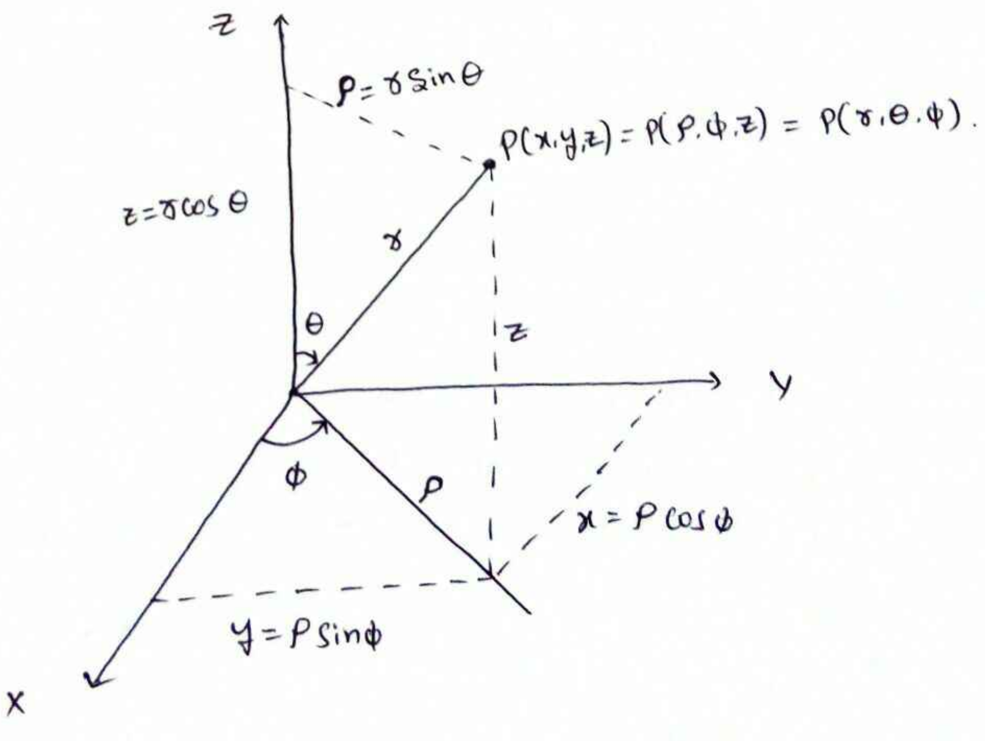
Properties:

$$a_r \cdot a_r = a_\theta \cdot a_\theta = a_\phi \cdot a_\phi = 1$$

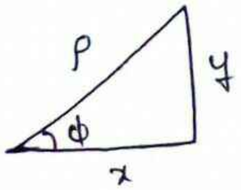
$$a_r \cdot a_\theta = a_\theta \cdot a_\phi = a_\phi \cdot a_r = 0$$

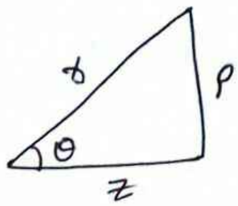
$$a_r \times a_\theta = a_\phi$$
$$a_\theta \times a_\phi = a_r$$
$$a_\phi \times a_r = a_\phi$$

Relation between variables (x, y, z) & (r, θ, ϕ):



from above figure





$\sin\phi = \frac{y}{\rho} \Rightarrow y = \rho \sin\phi$

$\cos\phi = \frac{x}{\rho} \Rightarrow x = \rho \cos\phi$

$\sin\theta = \frac{\rho}{r} \Rightarrow \rho = r\sin\theta$

$\cos\theta = \frac{z}{r} \Rightarrow z = r\cos\theta$

$\therefore$ Conversion from sperical to cartesian system is

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$

$\Rightarrow r^2 = \rho^2 + z^2 = x^2 + y^2 + z^2$

$\therefore r = \sqrt{x^2+y^2+z^2}$

$\Rightarrow \tan\theta = \frac{\rho}{z} = \frac{\sqrt{x^2+y^2}}{z}$

$\therefore \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$\Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right).$

$\therefore$ Conversion from cartesian to spherical system is

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}}$$

Problem

Given point $P(-2, 6, 3)$ and vector $\vec{A} = y\,a_x + (x+z)\,a_y$, express P and $\vec{A}$ in cylindrical and spherical coordinates.

Sol

At point P: $x = -2$, $y = 6$ & $z = 3$.

In cylindrical system $\rho = \sqrt{x^2+y^2} = \sqrt{4+36} = 6.32$

$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{-2}\right) = 108.43°$

$z = 3$

In spherical system:

$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+9} = 7$

$\theta = \tan^{-1} \frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\left(\frac{\sqrt{40}}{3}\right) = 64.62°$

$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{6}{-2}\right) = 108.43°$

$\therefore P(-2, 6, 3) = P(6.32, 108.43°, 3) = P(7, 64.62°, 108.43°)$.

$A = y\,a_x + (x+z)\,a_y = 6a_x + (-2+3)\,a_y$

$\therefore A = 6a_x + a_y$

3. VECTOR CALCULUS :

Differential Elements :

Differential elements in length, area and Volume are useful in vector calculus. They are defined in the cartesian, cylindrical and spherical coordinate systems.

A) Cartesian Coordinates :

i) Differential length $dl = dx\,a_x + dy\,a_y + dz\,a_z$

ii) Differential Surface area $dS = dy\,dz\,a_x$ (or)
$dx\,dz\,a_y$ (or)
$dx\,dy\,a_z$

iii) Differential volume $dv = dx\,dy\,dz$.

B) Cylindrical coordinates :

i) Differential length $dl = d\rho\,a_\rho + \rho\,d\phi\,a_\phi + dz\,a_z$

ii) Differential Surface area $dS = \rho\,d\phi\,dz\,a_\rho$ (or) $d\rho\,dz\,a_\phi$ (or) $\rho\,d\rho\,d\phi\,a_z$

iii) Differential volume $dv = \rho\,d\rho\,d\phi\,dz$.

C) Spherical Coordinates :

i) Differential length $dl = dr\,a_r + r\,d\theta\,a_\theta + r\sin\theta\,d\phi\,a_\phi$

ii) Differential Surface area $dS = r^2\sin\theta\,d\theta\,d\phi\,a_r$ (or)
$r\sin\theta\,dr\,d\phi\,a_\theta$ (or)
$r\,dr\,d\theta\,a_\phi$

iii) Differential volume $dv = r^2\sin\theta\,dr\,d\theta\,d\phi$

Del (∇) operator:

It is the vector differential operator and is defined as

→ In Cartesian coordinates $\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$

→ In cylindrical coordinates $\nabla = a_\rho \frac{\partial}{\partial \rho} + a_\phi \frac{\partial}{\rho \partial \phi} + a_z \frac{\partial}{\partial z}$

→ In Spherical coordinates $\nabla = \frac{\partial}{\partial r} a_r + \frac{1}{r}\frac{\partial}{\partial \theta} a_\theta + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} a_\phi$

Gradient of a scalar (∇V):

It is a vector quantity. It gives the maximum space - rate of change of the scalar.

Ex Temp. of soldering iron is scalar, but rate of change of Temp. is a vector.

→ In Cartesian coordinates $\nabla V = \frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z$

→ In cylindrical coordinates $\nabla V = \frac{\partial V}{\partial \rho} a_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} a_\phi + \frac{\partial V}{\partial z} a_z$

→ In Spherical coordinates $\nabla V = \frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} a_\phi$

Divergence of a Vector ($\nabla \cdot A$):

It is a scalar quantity. Divergence means the spreading (or) diverging of a quantity from a point.

Ex Sun rays ~~coming~~ coming from the Sun.

→ In cartesian coordinate $\nabla \cdot A = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$.

→ In cylindrical coordinates $\nabla \cdot A = \frac{\partial}{\partial \rho} A_\rho + \frac{1}{\rho} \frac{\partial}{\partial \phi} A_\phi + \frac{\partial}{\partial z} A_z$

→ In spherical coordinates $\nabla \cdot A = \frac{\partial}{\partial r} A_r + \frac{1}{r} \frac{\partial}{\partial \theta} A_\theta + \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} A_\phi$

Curl of a Vector ($\nabla \times A$) :

It is a vector quantity. It represents the rate of rotation of a vector quantity at a point.

It is defined as circulation per unit area.

→ In Cartesian coordinates $\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

→ In Cylindrical coordinates $\nabla \times A = \frac{1}{\rho} \begin{vmatrix} a_\rho & \rho a_\phi & a_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$

→ In Spherical coordinates $\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} a_r & r a_\theta & r\sin\theta \, a_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta A_\phi \end{vmatrix}$

Laplacian of a scalar ($\nabla^2 V$)

→ In cartesian coordinates $V = \nabla \cdot \nabla V = \nabla^2 V =$

$$\left[\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right] \cdot \left[\frac{\partial V}{\partial x} a_x + \frac{\partial V}{\partial y} a_y + \frac{\partial V}{\partial z} a_z\right]$$

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

→ In cylindrical coordinates $\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

→ In Spherical coordinates $\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2}$

Stoke's Theorem :

It states that the circulation of a vector field $\vec{A}$ around a closed Path L is equal to the surface integral of the curl of $\vec{A}$ over the open surface S bounded by "L"

(or)

A vector relation between the line and surface integrals is called stoke's theorem.

It is expressed as $$\boxed{\oint_L \vec{A}.dl = \int_S (\nabla \times \vec{A}).dS}$$

Divergence Theorem :

It States that the total outward flux of a vector field $\vec{A}$ through the closed Surface "S" is equal to the volume integral of the divergence $\vec{A}$.

(or)

The vector relation between the Surface and volume integrals is called divergence theorem.

It is expressed as $$\boxed{\oint_S \vec{A}.dS = \int_{Vol} (\nabla.A)\, dv}$$

UNIT-I

ELECTRIC FIELD

Coulomb's Law :

It is an experimental law formulated in 1785 by Charles Augustin de Coulomb. This law deals with the force a point charge exerts on another point charge.

"Coulumb's law states that the force between two point charges Q_1 & Q_2 seperated by a distance R is Proportional to the product of the charges and inversely proportional to the square of the distance b/w them."

→ The direction of the force is along the line joining the charges. Collomb also postulated that like charges repel and unlike charges attract with each other.

(or)

"Coulomb's law states that the force F between two point charges Q_1 and Q_2 is

* Along the line joining them
* Directly proportional to the product $Q_1 Q_2$ of the charges.
* Inversely proportional to the square of the distance R b/w them."

Expressed mathematically $$\boxed{F \propto \frac{Q_1 Q_2}{R^2} = K\frac{Q_1 Q_2}{R^2}} \xrightarrow{\text{Newtons (N)}} (1)$$

where K → Proportionality constant

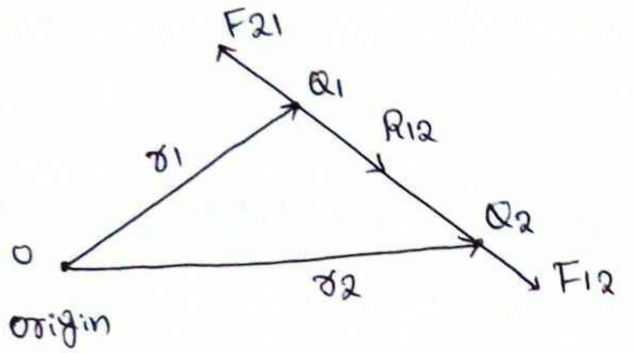
$$K = \frac{1}{4\pi\epsilon_0}$$

ϵ_0 → Permitivity of free space

$$\boxed{\epsilon_0 = 8.854\times10^{-12} \simeq \frac{10^{-9}}{36\pi}\ \text{F/m} \qquad K = \frac{1}{4\pi\epsilon_0} \simeq 9\times10^{9}\ \text{m/F}}$$

$$\boxed{\therefore\ F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}} \longrightarrow (2)$$

Coulomb vector force on point charges Q_1 & Q_2:



→ If point charges Q_1 & Q_2 are located at points having position vectors r_1 & r_2 then the force F_{12} on Q_2 due to Q_1 shown in figure is given by

$$\boxed{F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} a_{R12}} \longrightarrow (3)$$

from fig $r_1 + R_{12} - r_2 = 0 \Rightarrow R_{12} = r_2 - r_1$

and $R = |R_{12}|$

unit vector $\therefore a_{R12} = \frac{R_{12}}{|R_{12}|} = \frac{R_{12}}{R} \longrightarrow (4)$

Substituting eq (4) in (3)

$$\boxed{\therefore F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^3} R_{12} = \frac{Q_1 Q_2 (r_2 - r_1)}{4\pi\epsilon_0 |r_2 - r_1|^3}} \longrightarrow (5)$$

→ The force F_{21} on Q_1 due to Q_2 shown in above figure is given by $F_{21} = |F_{12}|\, a_{R21} = |F_{12}|\,(-a_{R12})$

$\therefore F_{21} = -F_{12}$ since $a_{R21} = -a_{R12}$.

Coulomb's Law:

It is an experimental law formulated in 1785 by Charles Augustin de Coulomb. This law deals with the force a point charge exerts on another point charge.

Coulomb's law states that the force F between two point charges Q_1 and Q_2 is

- → Along the line joining them
- → Directly proportional to the product $Q_1 Q_2$ of the charges
- → Inversely proportional to the square of the distance R b/w them.

$$F \propto \frac{Q_1 Q_2}{R^2}$$

$$\boxed{F = K \frac{Q_1 Q_2}{R^2}}$$

where K is proportionality constant

$$K = \frac{1}{4\pi\epsilon_0}$$

ϵ_0 is permitivity of free space.

$$k = \frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \text{ m/F}$$

$$\epsilon_0 = \frac{1}{36\pi \times 10^9} = 8.854 \times 10^{-12} \text{ F/m}$$

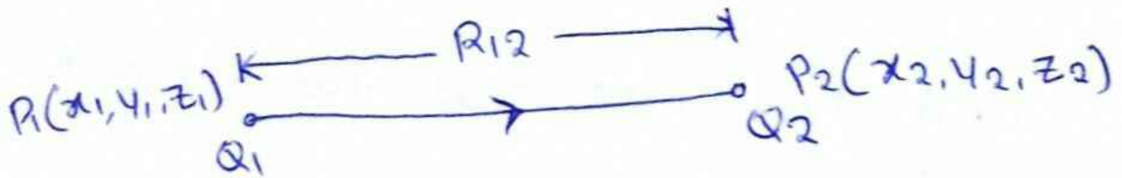
$$\therefore \boxed{F = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{R^2}}$$

The above force is a scalar quantity, becaus it has only magnitude doesnot have direction.

The vector force between two point charges can be computed from the formula

$$\boxed{F_{12} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{R_{12}^2} a_{R_{12}}}$$

where $a_{R_{12}} = \frac{R_{12}}{|R_{12}|}$ is a unit vector.



$$R_{12} = (x_2 - x_1)a_x + (y_2 - y_1)a_y + (z_2 - z_1)a_z$$

$$|R_{12}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

EF-2

→ If there are more than two point charges (or) For N Point Charges $Q_1, Q_2, \ldots Q_N$ located respectively, at Point with position vectors at $r_1, r_2, \ldots r_N$, the resultant force F on a charge Q located at point r is the vector Sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \ldots Q_N$. Hence

$$F = \frac{QQ_1(r-r_1)}{4\pi\epsilon_0|r-r_1|^3} + \frac{Q\cdot Q_2(r-r_2)}{4\pi\epsilon_0|r-r_2|^3} + \cdots + \frac{QQ_N(r-r_N)}{4\pi\epsilon_0|r-r_N|^3}$$

$$\therefore \boxed{F = \frac{Q}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{Q_k(r-r_k)}{|r-r_k|^3}}$$

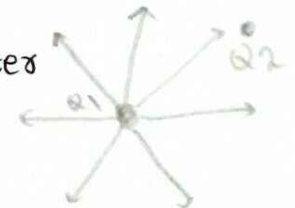
Electric Field Intensity (E):

It is defined as the force per unit charge when placed in an electric field. It is also called as electric field strength.

$$\boxed{E = \frac{F}{Q}}$$

N/C (or) Volts/meter

E is obviously in the direction of F.



→ Electric field intensity at point charge Q_2 due to point charge Q_1 is given by

$$E = \frac{F}{Q_2} = \frac{Q_1\cancel{Q_2}}{\cancel{Q_2}4\pi\epsilon_0 R^2}a_{R_{12}} = \frac{Q_1}{4\pi\epsilon_0}\frac{R_{12}}{R^3}$$

→ Electric field intensity at Point r due to point charge located at r' is given by

$$\boxed{E = \frac{Q}{4\pi\epsilon_0 R^2}a_R = \frac{Q(r-r')}{4\pi\epsilon_0|r-r'|^3}}$$

→ For N Point charges $Q_1, Q_2, \ldots Q_N$ located at $r_1, r_2, \ldots r_N$ the EFI at Point r is given by

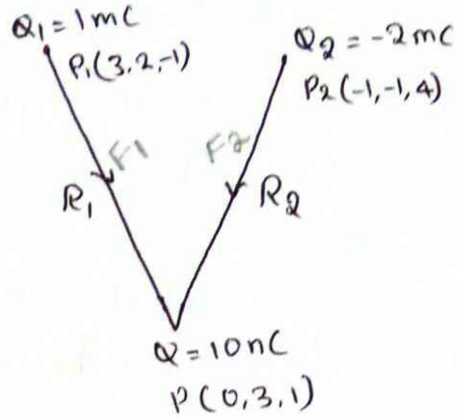
$$E = \frac{Q_1(r-r_1)}{4\pi\epsilon_0|r-r_1|^3} + \frac{Q_2(r-r_2)}{4\pi\epsilon_0|r-r_2|^3} + \cdots + \frac{Q_N(r-r_N)}{4\pi\epsilon_0|r-r_N|^3}.$$

$$\therefore \boxed{E = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{N}\frac{Q_k(r-r_k)}{|r-r_k|^3}}$$

Problem

Point charges 1 mC & −2 mC are located at (3, 2, −1) and (−1, −1, 4), respectively. Calculate the electric force on a 10 nC charge located at (0, 3, 1) and the electric field intensity at that point

Sol



$$F = \frac{Q}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{Q_k}{R^2} a_R \qquad \text{Here } k = 1,2$$

$$\therefore F = \frac{Q}{4\pi\epsilon_0}\left[\frac{Q_1 a_{R_1}}{R_1^2} + \frac{Q_2 a_{R_2}}{R_2^2}\right]$$

$$F = F_1 + F_2$$

$$R_1 = (0-3)a_x + (3-2)a_y + (1+1)a_z = -3a_x + a_y + 2a_z$$

$$|R_1| = \sqrt{9+1+4} = \sqrt{14}$$

$$R_2 = (0+1)a_x + (3+1)a_y + (1-4)a_z = a_x + 4a_y - 3a_z$$

$$|R_2| = \sqrt{1+16+9} = \sqrt{26}$$

$$\therefore F = \frac{Q}{4\pi\epsilon_0}\left[\frac{Q_1 R_1}{|R_1|^3} + \frac{Q_2 R_2}{|R_2|^3}\right] = 10\times10^{-9}\times9\times10^{9}\left[\frac{1\times10^{-3}(-3a_x+a_y+2a_z)}{14\sqrt{14}} + \frac{(-2\times10^{-3})(a_x+4a_y-3a_z)}{26\sqrt{26}}\right]$$

$$F = 9\times10\times10^{-3}\left[\frac{-3a_x+a_y+2a_z}{14\sqrt{14}} + \frac{(-2a_x-8a_y+6a_z)}{26\sqrt{26}}\right]$$

$$F = 9\times10^{-2}\left[\frac{-3a_x+a_y+2a_z}{14\sqrt{14}} + \frac{-2a_x-8a_y+6a_z}{26\sqrt{26}}\right]$$

$$\boxed{F = -6.507a_x - 3.817a_y + 7.506a_z \ \text{mN}}$$

Electric field Intensity at that point $E = \frac{F}{Q}$

$$E = (-6.507a_x - 3.817a_y + 7.506a_z) \times \frac{10^{-3}}{10\times10^{-9}}$$

$$\boxed{E = -650.7a_x - 381.7a_y + 750.6a_z \ \text{kV/m}}$$

Problem Point charges 5 nC & −2 nC are located at (2, 0, 4) & (−3, 0, 5), respectively.

(a) Determine the force on a 1 nC point charge located at (1, −3, 7).

(b) Find electric field E at (1, −3, 7).

Sol (a). $-1.004a_x - 1.284a_y + 1.4a_z$ nN & (b) $-1.004a_x - 1.284a_y + 1.4a_z$ V/m.

Electric fields due to Continuous Charge Distribution:

The force and Electric field charges occupying very small physical space.

It is also possible to have continuous charge distribution along a line, on a surface and in a volume (ρ_l, ρ_s & ρ_v).

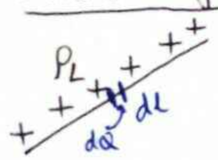
→ The charge element dQ and the total charge Q due to these charge distributions are obtained from the following figures.

Point charge:

$$E = \frac{Q}{4\pi\epsilon_0 R^2} a_R$$

$\rho_l = \frac{\text{Total charge}}{\text{Unit length}}$

Line charge:



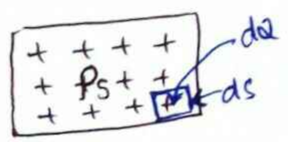
$$dQ = \rho_L \, dl$$

$$Q = \int_L \rho_l \, dl$$

$$E = \int_L \frac{\rho_l \, dl}{4\pi\epsilon_0 R^2} a_R$$

$\rho_s = \frac{\text{Total charge}}{\text{unit surface}}$

Surface charge:



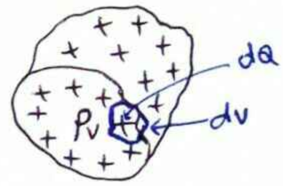
$$dQ = \rho_s \, ds$$

$$Q = \int_S \rho_s \, ds$$

$$E = \int_S \frac{\rho_s \, ds}{4\pi\epsilon_0 R^2} a_R$$

$\rho_v = \frac{\text{Total charge}}{\text{Unit volume}}$

Volume charge:



$$dQ = \rho_v \, dv$$

$$Q = \int_v \rho_v \, dv$$

$$E = \int_v \frac{\rho_v \, dv}{4\pi\epsilon_0 R^2} a_R$$

ρ_l, ρ_s & ρ_v are the charge densities.

A Line Charge:

→ Consider a line charge with uniform charge density ρ_l extented from A to B along the Z-axis as show in figure.

→ The charge element dQ associated with element $dl = dz$ of the line is

$$dQ = \rho_L \, dl = \rho_L \, dz$$

$$Q = \int_{z_A}^{z_B} \rho_L \, dz$$

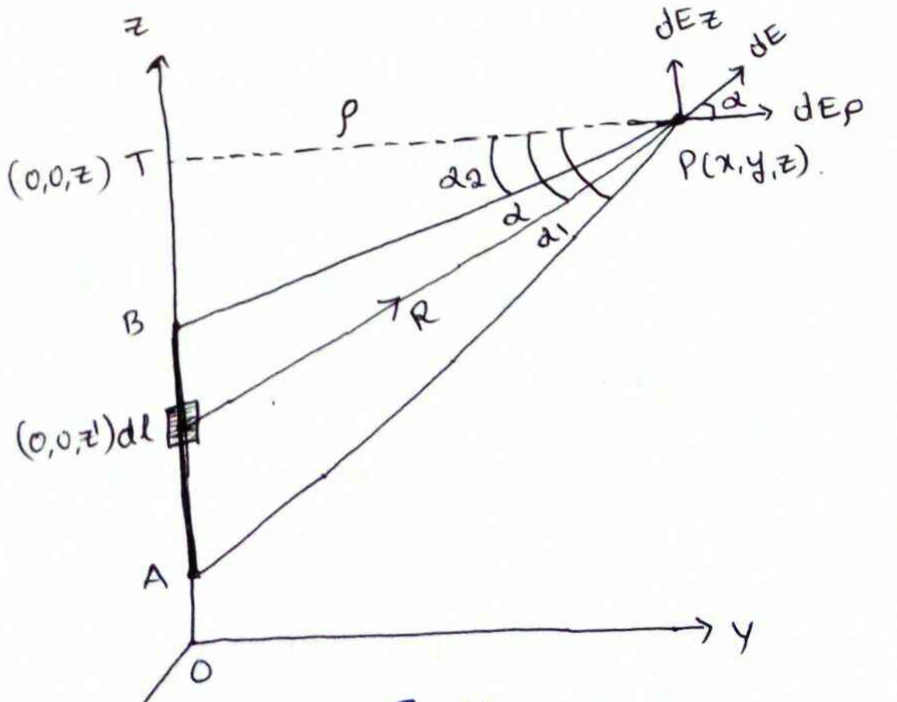


Fig (1).

→ The electric field intensity E at any arbitrary point $P(x, y, z)$ can be find by using the equation at $dl = dz'$

$$E = \int_L \frac{\rho_l \, dz}{4\pi\varepsilon_0 R^2} a_R \longrightarrow (1)$$

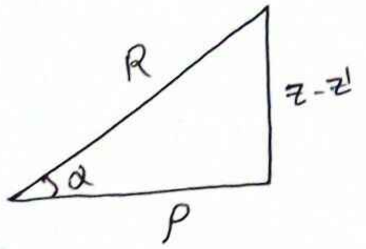
$$R = (x, y, z) - (0, 0, z')$$

$$R = x a_x + y a_y + (z - z') a_z \quad (\text{or})$$

from fig(1). $R = \rho a_\rho + (z - z') a_z$

$$|R| = \sqrt{\rho^2 + (z - z')^2}$$

from figure:



from(1) $E = \int_L \frac{\rho_l \, dz'}{4\pi\varepsilon_0 R^2} \frac{R}{|R|} = \int_L \frac{\rho_L \, dz' \, R}{4\pi\varepsilon_0 |R^3|}$

$$\sin\alpha = \frac{z - z'}{R} \Rightarrow z - z' = R\sin\alpha$$

$$\cos\alpha = \frac{\rho}{R} \Rightarrow \rho = R\cos\alpha$$

$$\tan\alpha = \frac{z - z'}{\rho} \Rightarrow z - z' = \rho\tan\alpha$$

$$E = \frac{\rho_L}{4\pi\varepsilon_0} \int \frac{\rho a_\rho + (z - z') a_z \, dz'}{[\rho^2 + (z - z')^2]^{3/2}}$$

$$E = \frac{\rho_l}{4\pi\varepsilon_0} \int \frac{R\cos\alpha \, a_\rho + R\sin\alpha \, a_z \, dz'}{[R^2\cos^2\alpha + R^2\sin^2\alpha]^{3/2}}$$

$$E = \frac{\rho_l}{4\pi\varepsilon_0} \int \frac{R[\cos\alpha \, a_\rho + \sin\alpha \, a_z] \, dz'}{[R^2[\cos^2\alpha + \sin^2\alpha]]^{3/2}}$$

$$\frac{R}{\rho} = \sec\alpha \Rightarrow R = \rho\sec\alpha$$

$$R^2 = \rho^2\sec^2\alpha$$

$$z' = OT - (z - z')$$

$$z' = OT - \rho\tan\alpha$$

$$dz' = 0 - \rho\sec^2\alpha \, d\alpha$$

$$dz' = -\rho\sec^2\alpha \, d\alpha$$

$$E = \frac{\rho_l}{4\pi\varepsilon_0} \int \frac{\cancel{R}[\cos\alpha \, a_\rho + \sin\alpha \, a_z] \, dz'}{R^{\cancel{3}2}}$$

$$E = \frac{\rho_L}{4\pi\varepsilon_0} \int \frac{(\cos\alpha \, a_\rho + \sin\alpha \, a_z)}{R^2} dz'$$

$$E = \frac{\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{(\cos\alpha \, a_\rho + \sin\alpha \, a_z)}{\rho^{\cancel{2}}\cancel{\sec^2\alpha}} (-\cancel{\rho}\cancel{\sec^2\alpha} \, d\alpha)$$

$$E = \frac{-\rho_L}{4\pi\varepsilon_0\rho} \int_{\alpha_1}^{\alpha_2} (\cos\alpha \, a_\rho + \sin\alpha \, a_z) \, d\alpha.$$

For finite line charge,

EF-4

$$E = \frac{-\rho_L}{4\pi\epsilon_0\rho}\left\{[+\sin\alpha]_{\alpha_1}^{\alpha_2} a_\rho + [\cos\alpha]_{\alpha_1}^{\alpha_2} a_z\right\}$$

$$E = \frac{-\rho_L}{4\pi\epsilon_0\rho}\left[+(\sin\alpha_2 - \sin\alpha_1) a_\rho + (\cos\alpha_2 - \cos\alpha_1) a_z\right]$$

$\cancel{E = \frac{\rho_L}{4\pi\epsilon_0\rho}}$ For an infinite line charge, Point B is at $(0,0,\infty)$ and A at $(0,0,-\infty)$ so that $\alpha_1 = \frac{\pi}{2}$ & $\alpha_2 = -\frac{\pi}{2}$; the z-vanishes

$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[-(\sin(-\tfrac{\pi}{2}) - \sin(\tfrac{\pi}{2})) a_\rho + (\cos(-\tfrac{\pi}{2}) - \cos(\tfrac{\pi}{2})) a_z\right]$$

$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[-(-\sin\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}) a_\rho + (-\cos\tfrac{\pi}{2} - \cos\tfrac{\pi}{2}) a_z\right]$$

$$E = \frac{\rho_L}{4\pi\epsilon_0\rho}\left[-(-1-1) a_\rho + (-0-0) a_z\right]$$

$$E = \frac{\rho_L}{\cancel{2}\,4\pi\epsilon_0\rho}\left[\cancel{2} a_\rho\right]$$

$$\boxed{E = \frac{\rho_L}{2\pi\epsilon_0\rho} a_\rho} \rightarrow (2)$$

Infinite line charge

$z' = OT - \rho\tan\alpha$

(i) $\infty = OT - \rho\tan\alpha_2$

$\alpha_2 = \tan^{-1}(-\infty) = -\frac{\pi}{2}$

(ii) $-\infty = OT - \rho\tan\alpha_1$

$\alpha_1 = \tan^{-1}(\infty) = \frac{\pi}{2}$

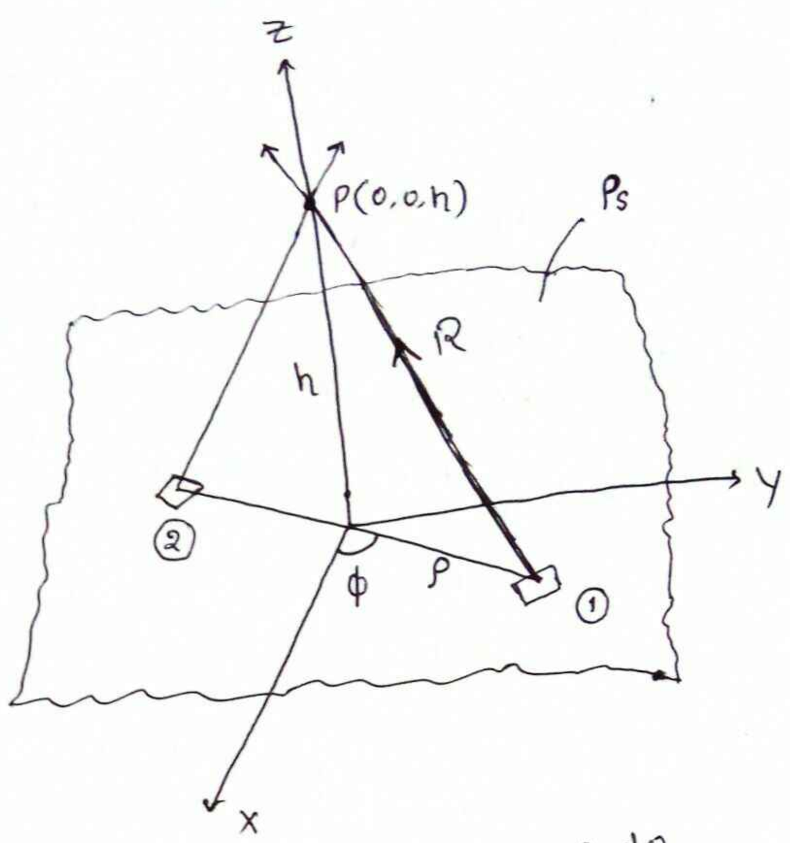
Eq(2) is obtained for an infinite line charge along the z-axis so that ρ & a_ρ have their usual meaning. If the line is not along the z-axis ρ is the perpendicular distance from the line to the point of interest and a_ρ is a unit vector along that distance directed from the line charge to the field point.

A Surface Charge :

Consider an infinite sheet of charge in the x-y plane with uniform charge density ρ_S. The charge associated with an elemental area dS is

$$dQ = \rho_S \, dS$$

$$Q = \int_S \rho_S \, dS$$



→ The electric field intensity E at Point P(0,0,h) by the charge dQ on the elemental Surface 1. shown in figure.

$$E = \int \frac{Q}{4\pi\epsilon_0 R^2} a_R = \int \frac{\rho_S \, dS}{4\pi\epsilon_0 R^2} a_R \qquad \text{(or)} \quad dE = \frac{dQ}{4\pi\epsilon_0 R^2} a_R$$

from fig. $\rho + R - h = 0 \Rightarrow R = -\rho + h$

$$\therefore R = -\rho a_\rho + h a_z$$

$$|R| = \sqrt{\rho^2 + h^2} \quad \& \quad a_R = \frac{R}{|R|}$$

$dS = \rho \, d\rho \, d\phi$ [$\because$ Surface area along z-axis in cylindrical coordinate System].

Substitute these values in the above equation.

$$\therefore E = \int \frac{\rho_S \, dS \cdot R}{4\pi\epsilon_0 |R|^3} = \int \frac{\rho_S \, \rho d\rho d\phi \cdot (-\rho a_\rho + h a_z)}{4\pi\epsilon_0 \left(\sqrt{\rho^2 + h^2}\right)^3}$$

$$E = \frac{\rho_S}{4\pi\epsilon_0} \int \frac{(-\rho a_\rho + h a_z) \, \rho \, d\rho \, d\phi}{(\rho^2 + h^2)^{3/2}}$$

EF-5

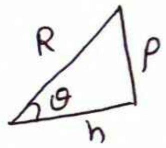
→ Due to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2, whose contribution along a_ρ cancels that of element 1. is shown in figure.

Thus the contribution to E_ρ add up to zero so that E has only z-component.

$$\therefore E = \frac{\rho_s}{4\pi\epsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\rho\, d\rho\, d\phi}{(\rho^2+h^2)^{3/2}}\, a_z$$

$$E = \frac{\rho_s h}{\cancel{4\pi}\epsilon_0}\,\cancel{2\pi}\int_0^{\infty}\frac{\rho\, d\rho}{[\rho^2+h^2]^{3/2}}\, a_z$$

from fig



$\tan\theta = \rho/h$

$\rho = h\tan\theta$

$d\rho = h\sec^2\theta\, d\theta$

when $\rho=0$; $\tan\theta = 0$
$\theta = \tan^{-1}(0) = 0$

when $\rho=\infty$; $\tan\theta = \infty$
$\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$

$$E = \frac{\rho_s h}{2\epsilon_0}\int_0^{\pi/2}\frac{h\tan\theta\, h\sec^2\theta\, d\theta}{(h^2\tan^2\theta+h^2)^{3/2}}\, a_z$$

$$E = \frac{\rho_s h}{2\epsilon_0}\int_0^{\pi/2}\frac{h^2\tan\theta\sec^2\theta\, d\theta}{[h^2(\tan^2\theta+1)]^{3/2}}\, a_z$$

$$E = \frac{\rho_s \cancel{h}}{2\epsilon_0}\int_0^{\pi/2}\frac{\cancel{h^2}\tan\theta\, \cancel{\sec^2}\theta\, d\theta}{\cancel{h^3}\, \sec^{\cancel{3}}\theta}\, a_z$$

$$E = \frac{\rho_s}{2\epsilon_0}\int_0^{\pi/2}\frac{\tan\theta\, d\theta}{\sec\theta}\, a_z = \int_0^{\infty}\frac{\rho_s}{2\epsilon_0}\frac{\sin\theta}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\, d\theta\, a_z$$

$$E = \frac{\rho_s}{2\epsilon_0}\int_0^{\pi/2}\sin\theta\, d\theta\, a_z = \frac{\rho_s}{2\epsilon_0}\left[-\cos\theta\right]_0^{\pi/2} a_z = \frac{\rho_s}{2\epsilon_0}(-(-1))\, a_z = \frac{\rho_s}{2\epsilon_0}\, a_z.$$

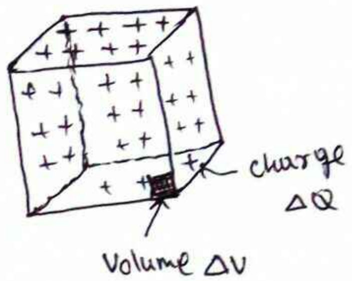
$$\boxed{\therefore E = \frac{\rho_s}{2\epsilon_0}\, a_z}$$

E has only z-component if the charge is in the x-y plane.

→ In general, for an infinite sheet of charge $\boxed{E = \frac{\rho_s}{2\epsilon_0}\, a_n}$

A Volume Charge :

A charge uniformly distributed through out a volume is called a volume charge. The volume charge density ρ_v is defined as the charge per unit volume. $\boxed{\rho_v = \frac{Q}{V}}$



When charge Q is distributed throughout a specified volume V, then each element contributes electric field with respect to some external point.

Consider a small amount of charge ΔQ over a small volume ΔV, the ratio $\frac{\Delta Q}{\Delta V}$ is called volume charge density. It is denoted with ρ_v & measured in C/m^3.

$$\rho_v = \frac{\Delta Q}{\Delta V} = \frac{dQ}{dv} \Rightarrow dQ = \rho_v \, dv$$

$$\Rightarrow \boxed{Q = \int \rho_v \, dv}$$

→ Electric field induced due to a point charge is

$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} a_R$$

→ Electric field induced due to a small charge is

$$dE = \frac{1}{4\pi\epsilon_0} \frac{dQ}{R^2} a_R = \frac{1}{4\pi\epsilon_0} \rho_v dv . a_R$$

$$\boxed{E = \frac{1}{4\pi\epsilon_0} \int_v \rho_v dv . a_R}$$

* A small charge Q occupying very small area (or) volume is called a point charge.
* A charge uniformly distributed along a line is called line charge.
* A charge uniformly distributed over a surface (or) sheet is called surface charge.
* A charge uniformly distributed throughout a volume is called volume charge.

Problem EF-6

A circular ring of radius a carries a uniform charge ρ_L c/m and is placed on the xy-plane with axis same as the z-axis.

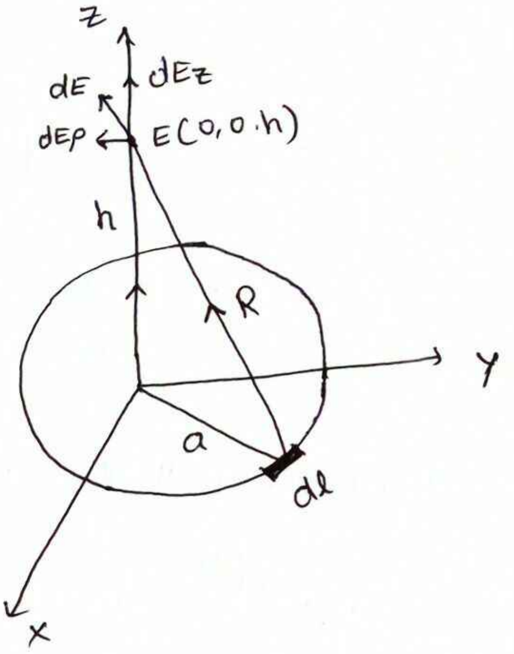
(a) Show That $$E(0,0,h) = \frac{\rho_L a h}{2\epsilon_0[h^2+a^2]^{3/2}} a_z$$

(b) What values of h gives the maximum value of E ?

(c) If the total charge on the ring is Q, find E as $a \to 0$.

Sol

(a) For a line charge the electric field intensity is given by



$$E = \int_L \frac{\rho_L dl}{4\pi\epsilon_0 R^2} a_R .$$

from fig: $R + a - h = 0$

$R = -a + h$

$\therefore R = a(-a_\rho) + h a_z$

$|R| = \sqrt{a^2+h^2}$

$a_R = \frac{R}{|R|}$ & $dl = a\,d\phi$.

Substitute all these values in the above equation.

$$E = \int \frac{\rho_L}{4\pi\epsilon_0} \frac{(-a a_\rho + h a_z)}{[\sqrt{a^2+h^2}]^3} a\,d\phi$$

$$E = \frac{\rho_L}{4\pi\epsilon_0} \int_{\phi=0}^{2\pi} \frac{(-a a_\rho + h a_z)}{(a^2+h^2)^{3/2}} a\,d\phi$$

→ By symmetry, the contribution along a_ρ added up to zero. This is for every element dl there is a corresponding element diametrically opposit to it. So that the two contributions cancel each other. Thus E has only z-component.

that is, $E = \frac{\rho_L a h a_z}{4\pi\epsilon_0 (a^2+h^2)^{3/2}} \int_0^{2\pi} d\phi = \frac{\rho_L a h a_z}{\underset{2}{4\pi}\epsilon_0 (a^2+h^2)^{3/2}} \times 2\pi$

$$\boxed{\therefore E(0,0,h) = \frac{\rho_L a h}{2\epsilon_0 (h^2+a^2)^{3/2}} a_z}$$

(b)

$$\frac{d|E|}{dh} = \frac{\rho_L a}{2\epsilon_0} a_z \frac{d}{dh} \frac{h}{(h^2+a^2)^{3/2}}$$

$$= \frac{\rho_L a}{2\epsilon_0} a_z \left[\frac{(h^2+a^2)^{3/2}\cdot(1) - h\cdot\frac{3}{\cancel{2}}(h^2+a^2)^{1/2}\cdot \cancel{2}h}{((h^2+a^2)^{3/\cancel{2}})^{\cancel{2}}} \right]$$

$$= \frac{\rho_L a}{2\epsilon_0} a_z \left[\frac{(h^2+a^2)^{3/2} - 3h^2(h^2+a^2)^{1/2}}{(h^2+a^2)^3} \right]$$

For maximum value of E $\frac{d|E|}{dh} = 0$ i.e

$(h^2+a^2)^{3/\cancel{2}} = 3h^2 (h^2+a^2)^{\cancel{1/2}} \Rightarrow h^2+a^2 = 3h^2 \Rightarrow a^2 = 2h^2$

$\therefore h^2 = \frac{a^2}{2} \Rightarrow \boxed{h = \pm\frac{a}{\sqrt{2}}}$

(c) Since the charge is uniformly distributed, the line charge density is $\rho_L = \frac{Q}{l} = \frac{Q}{2\pi a}$ ∴ Circumference of the circle $= 2\pi r$

$$E = \frac{\rho_L a h}{2\epsilon_0 (h^2+a^2)^{3/2}} a_z = \frac{Q \cancel{a} h}{2\pi \cancel{a}\, 2\epsilon_0 (h^2+a^2)^{3/2}} a_z$$

$$E = \frac{Qh}{4\pi\epsilon_0 (h^2+a^2)^{3/2}} a_z$$

As $a \to 0$; $E = \frac{Q\cancel{h}}{4\pi\epsilon_0 h^{\cancel{3}2}} a_z = \frac{Q}{4\pi\epsilon_0 h^2} a_z$.

In general $\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2} a_R}$

Electric Flux & Flux Density:

Consider two Point charges placed in a space. If one charge is brought near another charge, a force is exerted b/w them. An experiment conducted by Michael Faraday in 1837 Showed that if a positive charge is placed near a negative charge, due to the force, an elecric field is developed between the charges. The lines of force originated radially from positive charge and terminated at negative charge as shown in below figure.

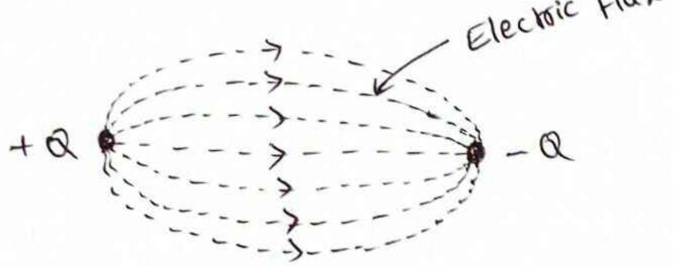


fig: Electric field b/w 2 Point charges.

Electric flux (ψ):

The total number of Lines of force in an electric field is called electric flux. The lines of force are called flux lines (or) Stream Lines.

The electric harge produces the electric flux and as the charge increases the elecric flux also increases. there fore the electric flux is numerically equal to the electric charge. If an electric charge produces Q coulombs, then the electric flux is given by $\boxed{\psi = Q \text{ coulombs}}$

Note:

1. The flux Lines are always Parallel to each other
2. The flux lines do not depend on the medium in which the charges are placed.
3. The flux lines radiate in all directions from/into the charges.

Electric Flux Density ($\vec{D}$):

Consider the electric flux ψ generated from Q coulombs of electric charges as shown in figure.

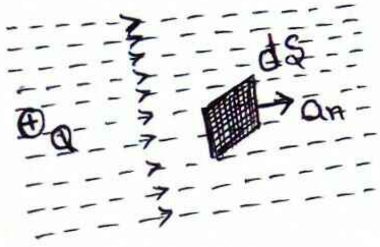


fig: Electric flux lines.

→ Let the differential surface area dS at point perpendicular to the lines of force. The electric flux crossing through this surface-area is $d\psi$.

* Def: The net electric flux passing through the unit surface area is called electric flux density $\vec{D}$.

→ The direction of electric flux density is normal to the surface-area. It is expressed as $\boxed{\vec{D} = \frac{d\psi}{dS}\,\vec{a}_n}$ C/m^2

Electric flux density (EFD) due to point charge Q:

→ Consider an imaginary sphere of radius 'r' and a point charge +Q is placed at the centre of the sphere as show in figure.

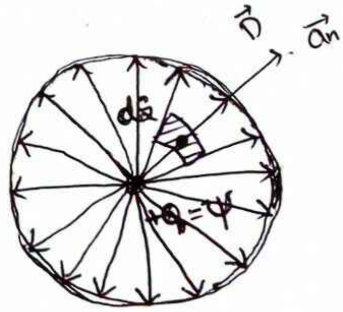


fig: flux from point charge

→ The flux lines originates from the point charge distributes radially over the surface of the sphere.

The flux density at the differential surface dS is

$$D = \frac{d\psi}{dS}\,a_n$$

→ If the total surface area is considered through which the flux ψ is passing, the flux density is $D = \frac{\text{Total flux}}{\text{Surface area of sphere}}$

$$D = \frac{\psi}{4\pi r^2}\,a_n = \frac{Q}{4\pi r^2}\,a_n.$$

Since $a_n = a_r$, radial direction normal to the surface area.

then $$\boxed{D = \frac{Q}{4\pi r^2} a_r}$$

The electric flux density is also called displacement flux density.

Relation b/w Electric field Intensity & Electric flux Density :

Electric field intensity $\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} a_r$

$$E\epsilon_0 = \frac{Q}{4\pi r^2} a_r \longrightarrow (1)$$

Electric flux density $\vec{D} = \frac{Q}{4\pi r^2} a_r \longrightarrow (2)$

equatin eq (1) & (2) then $\boxed{\vec{D} = \vec{E}\epsilon_0}$

$\therefore$ $\vec{D}$ & $\vec{E}$ are related through the permitivity of the medium.

Electric flux Density Due to Line charge :

→ for infinite line charge density ρ_L c/m

we know that $E = \frac{\rho_L}{2\pi\epsilon_0\rho} a_\rho$

then $D = \epsilon_0 E = \epsilon_0 \frac{\rho_L}{2\pi\epsilon_0\rho} a_\rho = \frac{\rho_L}{2\pi\rho} a_\rho$

$\therefore$ The flux density is $$\boxed{\vec{D} = \frac{\rho_L}{2\pi\rho} a_\rho}$$

Electric flux density due to Surface (Sheet) charge :

→ for an infinite sheet charge density ρ_s C/m^2

we know that $E = \frac{\rho_s}{2\epsilon_0} a_n$

then $D = \epsilon_0 E = \epsilon_0 \frac{\rho_s}{2\epsilon_0} a_n = \frac{\rho_s}{2} a_n$.

∴ The flux density is $\boxed{\vec{D} = \frac{\rho_s}{2} a_n}$

Electric flux density due to volume charge :

→ for a volume charge density ρ_v C/m^3

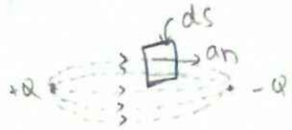
we know that $E = \frac{1}{4\pi\epsilon_0} \int_v \rho_v \, dv \, a_r$.

then $D = \epsilon_0 E = \epsilon_0 \cdot \frac{1}{4\pi\epsilon_0 r^2} \int_v \rho_v \, dv \, a_r$

The flux density $\boxed{D = \frac{1}{4\pi r^2} \int_v \rho_v \, dv \, a_r}$

Gauss's Law :

Gauss's law states that the total electric flux ψ through any closed surface is equal to the total charge enclosed by that surface.



$$\boxed{\nabla \cdot D = \rho_v}$$

$ds = ds\,a_n$

$D = \frac{d\psi}{ds} \Rightarrow d\psi = D\,ds\cos\theta = D \cdot ds$

D is normal to the surface
ds direction is normal to the surface
$\therefore$ angle b/w D & ds is zero; $\theta = 0$
i.e. D & ds are parallel
Hence $d\psi = D\,ds = D \cdot ds$

Proof: $\psi = Q_{enclosed}$

→ From electric flux density $D = \frac{d\psi}{ds}\, a_n$

$$d\psi = D \cdot ds \Rightarrow \psi = \oint_S D \cdot ds$$

→ Applying divergence theorem to the above equation

$$\psi = \int_S D \cdot ds = \int_V \nabla \cdot D \, dv$$

→ Total charge enclosed by the volume is

$$Q = \int_V \rho_v \, dv$$

$\therefore$ from the statement $$\boxed{\psi = Q = \int_S D \cdot ds = \int_V \nabla \cdot D \, dv = \int_V \rho_v \, dv}$$

It is the integral form of Gauss's law

→ The point form (or) differential form of Gauss law is $\boxed{\nabla \cdot D = \rho_v}$

→ The above equation states that the volume charge density is same as the divergence of the electric flux density.

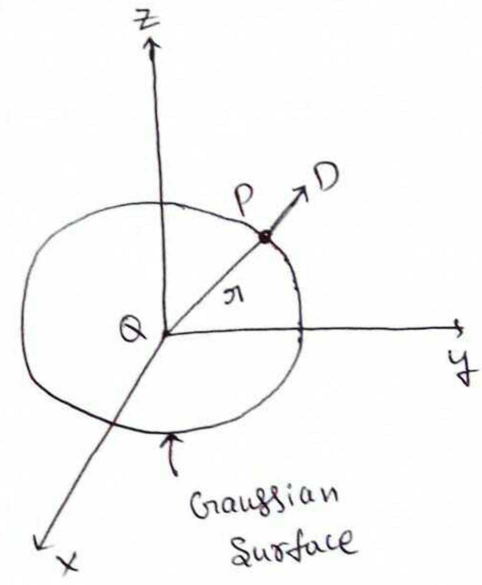
* Gauss's law is used for only symmetrical charge distributions. It is the limitation of Gauss's law.

Applications of Gauss's law :

→ Gauss's law is used to compute electric field intensity (E) and electric flux density (D) due to symmetric charge distributions.

* The Gaussian surface is always a closed surface. It may be of any shape.
* The surface is choosen such that at each point on the Gaussian surface, flux density D is either normal (or) tangential to the surface.
  ⇒ when D is normal to the Gaussian surface $\boxed{D.dS = D\,dS}$ because D is constant on the surface
  ⇒ when D is tangential $\boxed{D.dS = 0.}$

1. Point charge :



→ Suppose a point charge Q is located at the origin. To determine D at a point P, it is easy to see that choosing a spherical surface containing P will satisfy symmetry conditions

→ Thus a spherical surface centered at the origin is the Gaussian surface is shown in figure.

→ Since D is everywhere normal to the Gaussian surface, i.e.,

$$D = D_r a_r$$

applying Gauss's law

$$\psi = Q_{enclosed} = \oint_S D.dS = D_r \oint_S dS$$

dS → Surface[charge] area along r-direction in spherical coordinate system.

$dS = r^2 \sin\theta\, d\theta\, d\phi\, a_r$ & $\bar{D} = D_r a_r$; $\oint_S D.dS = \oint_S D_r . r^2 \sin\theta\, d\theta\, d\phi\, (a_r . a_r)$

$$\therefore \Psi = Q = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

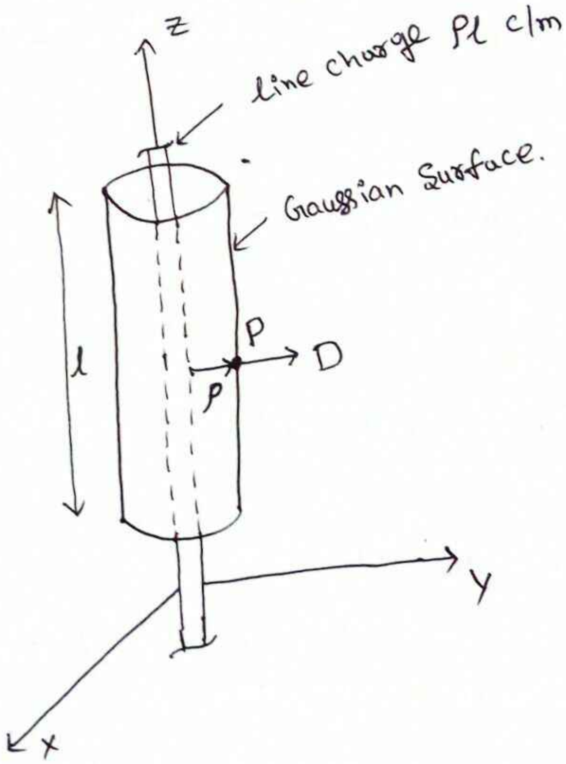
$$Q = D_r \, 2\pi \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta$$

$$Q = D_r \, 2\pi r^2 \left[-\cos\theta\right]_0^{\pi} = D_r \, 2\pi r^2 \left[-(-1-1)\right]$$

$$Q = 4\pi r^2 D_r. \Rightarrow D_r = \frac{Q}{4\pi r^2}$$

$$\therefore D = D_r a_r \Rightarrow \boxed{D = \frac{Q}{4\pi r^2} a_r}$$

2. Infinite Line Charge :



→ Suppose the infinite line of uniform charge ρ_L c/m lies along the z-axis.

→ To determine D at a point P, we choose a cylindrical surface containing P to satisfy the symmetry condition as shown in figure.

→ The electric flux density D is constant on and normal to the cylindrical gaussian surface; i.e. $D = D_\rho a_\rho$

applying Gauss's law

$$\Psi = Q_{enclosed} = \oint_S D \cdot dS = D_\rho \oint_S dS$$

dS → Surface area along ρ direction in cylindrical coordinate system.

$$\therefore \psi = Q = D_\rho \int_{z=0}^{l} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz$$

$$Q = D_\rho \, \rho . l \, 2\pi = 2\pi \rho l D_\rho \longrightarrow (1)$$

→ Total line charge is $dQ = \rho_L \, dl$

$$Q = \int_L \rho_L \, dl = \rho_L . l \longrightarrow (2)$$

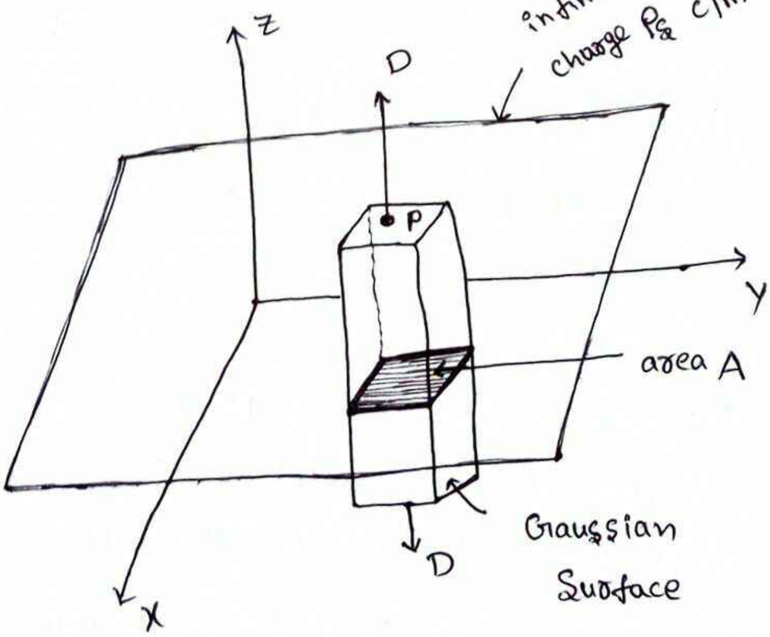
Equating eq (1) & (2) $2\pi \rho \cancel{l} D_\rho = \rho_L . \cancel{l}$

$$\boxed{D_\rho = \frac{\rho_L}{2\pi \rho}}$$

$$\therefore D = D_\rho a_\rho \Rightarrow \boxed{D = \frac{\rho_L}{2\pi\rho} a_\rho}$$

→ Electric field Intensity $E = \frac{D}{\epsilon_0} \Rightarrow \boxed{E = \frac{\rho_L}{2\pi \epsilon_0 \rho} a_\rho}$

3. Infinite Sheet of charge :



→ Consider the infinite sheet of uniform charge ρ_s C/m² lying on the z=0 plane.

→ To determine D at point P, we choose a rectangular box that is cut Symmetrically by the sheet of charge and has two of its faces parallel to the sheet as shown in figure.

→ As D is normal ($\perp$er) to the sheet i.e., $D = D_z a_z$

applying Gauss law

$$\Psi = Q_{enclosed} = \oint_S D \cdot dS$$

$$Q = D_z \left[\int_{top} dS + \int_{bottom} dS \right]$$

$$Q = D_z [A + A] = D_z 2A \longrightarrow (1)$$

→ total Surface charge $dQ = \rho_S \, dS$

$$Q = \rho_S \int dS = \rho_S A \longrightarrow (2)$$

$\int dS = \int dx\, dy = xy = A$

area of rectangle

$A = a \times b$

Equating eq (1) & (2)

$$D_z 2\cancel{A} = \rho_S \cancel{A}$$

$$D_z = \frac{\rho_S}{2}$$

$$D = D_z a_z \Rightarrow \boxed{D = \frac{\rho_S}{2} a_z}$$

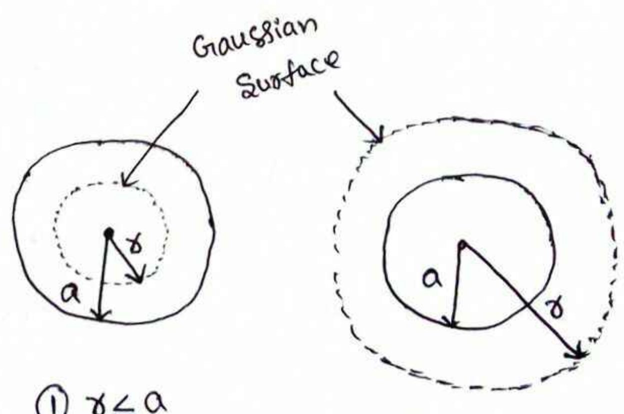
→ Electric Field Intensity $E = \frac{D}{\epsilon_0} \Longrightarrow \boxed{E = \frac{\rho_S}{2\epsilon_0} a_z}$

4. Uniformly charged Sphere :

→ Consider a sphere of radius a with a uniform charge ρ_v c/m³.

→ To determine D everywhere we construct Gaussian Surface for cases $r \leq a$ & $r \geq a$ seperately.

→ Since the charge has spherical Symmetry, it is obvious that a sphorical Surface is an appropriate Gaussian Surface.



① $r \leq a$ ② $r \geq a$

Case ①: for $r \leq a$, the total charge enclosed by the Spherical Surface of radius r, as shown in figure.

$$Q_{enclosed} = \int \rho_v \, dv = \rho_v \int dv$$

$$Q = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

∵ differential volume in spherical coordinate system $r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$Q = \rho_v \, 2\pi \left[-\cos\theta\right]_0^{\pi} \frac{r^3}{3} = \rho_v \, 2\pi (2) \frac{r^3}{3}$$

$$\boxed{\therefore Q = \rho_v \frac{4}{3} \pi r^3}$$

flux $\Psi = \oint_S D \cdot dS = D_r \oint_S dS$

$D = D_r a_r$

Surface area along r-direction

$dS = r^2 \sin\theta \, d\theta \, d\phi \, a_r$.

$$\Psi = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$\Psi = D_r \, 2\pi \left[-\cos\theta\right]_0^{\pi} r^2 = \boxed{D_r \, 4\pi r^2 = \Psi}$$

$$\therefore Q_{enclosed} = \Psi \Rightarrow \rho_v \frac{4}{3} \cancel{\pi} r^{\cancel{3}} = D_r \cancel{4\pi} \cancel{r^2} \Rightarrow D_r = \frac{\rho_v r}{3}$$

$$\therefore D = D_r a_r \Rightarrow \boxed{D = \frac{r}{3} \rho_v a_r} \longrightarrow (1)$$

Case ②: for $r \geq a$ the ~~total~~ charge enclosed by the surface is entire charge

$$Q = \int \rho_v \, dv = \rho_v \int dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta \, dr \, d\theta \, d\phi = \rho_v \frac{4}{3} \pi a^3$$

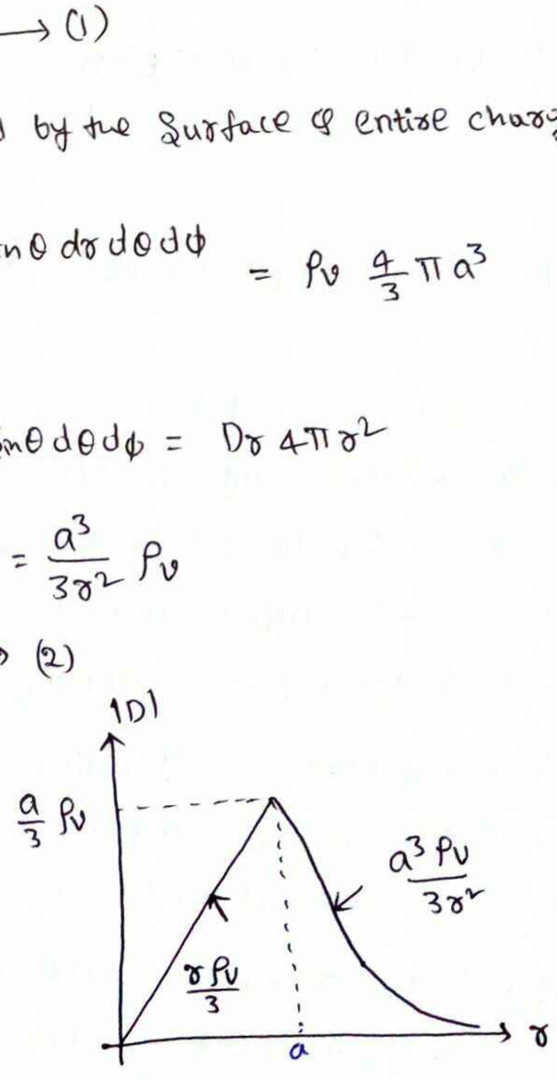
$$\Psi = \oint_S D \cdot dS = D_r \int dS = D_r \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = D_r \, 4\pi r^2$$

$$\therefore Q = \Psi \Rightarrow \rho_v \frac{4}{3} \pi a^3 = D_r \, 4\pi r^2 \Rightarrow D_r = \frac{a^3}{3r^2} \rho_v$$

$$\therefore D = D_r a_r \Rightarrow \boxed{D = \frac{a^3}{3r^2} \rho_v a_r} \longrightarrow (2)$$

→ from eq (1) & (2), D every where is given by

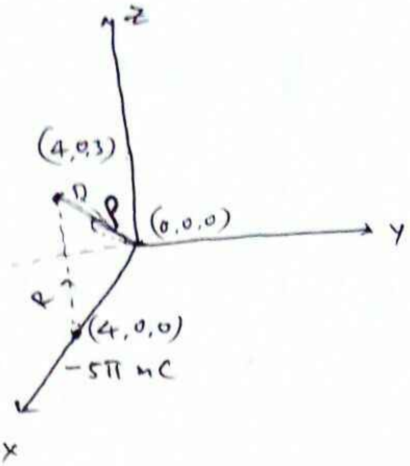
$$D = \begin{cases} \frac{r}{3} \rho_v a_r & 0 < r \leq a \\ \frac{a^3}{3r^2} \rho_v a_r & r \geq a \end{cases}$$



|D| verses r for a uniform charge sphere.

EF – 11.1

4.7 Determine D at (4,0,3) if there is a point charge -5π mC at (4,0,0) and a line charge 3π mC/m along the y-axis



Sol Electric flux density due to point charge

$$D_Q = \epsilon_0 E = \epsilon_0 \frac{Q}{4\pi\epsilon_0 R^2} a_R = \frac{Q}{4\pi R^2} a_R = \frac{Q}{4\pi |R|^3} R$$

$$R = (4a_x - 4a_x) + (0a_y - 0a_y) + (3a_z - 0a_z)$$

$$R = 0 + 0 + 3a_z = 3a_z$$

$$|R| = \sqrt{3^2} = \sqrt{9} = 3$$

$$D = \frac{-5\pi \times 10^{-3}}{4\pi |3|^3}(3a_z) = \frac{-5 \times 10^{-3}\, 3a_z}{4 \times 27} = -0.138\, a_z \text{ mC/m}^2$$

Electric flux density due to line charge

$$D_L = \frac{\rho_L}{2\pi\rho} a_\rho = \frac{\rho_L}{2\pi |\rho|^2} \rho$$

$$\rho = 4a_x + 0 + 3a_z = 4a_x + 3a_z$$

$$|\rho| = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$$

$$\therefore D_L = \frac{3\pi \times 10^{-3}}{2\pi |5|^2} \times (4a_x + 3a_z) = 0.24a_x + 0.18a_z \text{ mC/m}^2$$

$$\boxed{\therefore D = D_Q + D_L = 240a_x + 42a_z \ \mu\text{C/m}^2}$$

4.8 Given that $D = z\rho \cos^2\phi\, a_z$ C/m², calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1m [redacted] with $-2 \le z \le 2$ m

Sol $D = z\rho\cos^2\phi\, a_z$ C/m²

from Gauss law $\rho_v = \nabla \cdot D = \left(\frac{d}{dx}a_x + \frac{d}{dy}a_y + \frac{d}{dz}a_z\right)\cdot(z\rho\cos^2\phi\, a_z) = \frac{d}{dz} z\rho\cos^2\phi$

$$\rho_v = 1\,\rho\cos^2\phi = \rho\cos^2\phi \ \ ?$$

$(\rho\ \phi\ z)$

at $(1, \pi/4, 3)$. $\rho_v = 1 \cdot \cos^2(\pi/4) = 1\cdot(0.5) = 0.5$ C/m³ | $\cos(\pi/4) = \frac{1}{\sqrt{2}}$

Method I

$$Q = \int \rho_v\, dv = \int_v \rho\cos^2\phi\ \rho\, d\rho\, d\phi\, dz$$

| volume of the cylinder $dv = \rho\, d\rho\, d\phi\, dz$

$$Q = \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2\phi\, d\phi \int_{\rho=0}^{1} \rho^2 d\rho = [z]_{-2}^{2}\left(\frac{\rho^3}{3}\right)_0^1 \int_{\phi=0}^{2\pi}\left(\frac{1+\cos 2\pi}{2}\right) d\phi$$

$$Q = 4 \cdot \frac{1}{3} \cdot \left[\int_0^{2\pi} \frac{1}{2} d\phi + \int_0^{2\pi} \frac{\cos 2\pi}{2} d\phi \right] = \frac{4}{3} \left[\frac{\cancel{2}\pi}{\cancel{2}} + \frac{(\sin 4\pi - \sin 0)^{0}}{2} \right]$$

$$Q = \frac{4\pi}{3} C$$

$$\boxed{\therefore Q = \frac{4\pi}{3} C}$$

method 2 from Gauss Law

$$Q = \Psi = \oint D \cdot ds = \left[\int_s + \int_t + \int_b \right] D \cdot ds = \Psi_s^{\to 0} + \Psi_t + \Psi_b$$

$$\Psi_t = \int_t D \cdot ds = \int_t z\rho \cos^2\phi \, a_z \cdot \rho \, d\rho \, d\phi \Big|_{z=2}$$

surface of the cylinder along z-axis
$ds = \rho d\rho d\phi$ (top)

$$\Psi_t = 2 \int_{\rho=0}^{1} \rho^2 d\rho \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi = 2\left(\frac{1}{3}\right) \cdot \left(\frac{2\pi}{2} + 0\right) = \frac{2}{3}\pi$$

$$\therefore \Psi_t = \frac{2\pi}{3}$$

4.12 + 2 = ✓
6.12

Ψ_b. $ds = -\rho d\rho d\phi$ (bottom)

$$\Psi_b = \int_b D \cdot ds = \int_b z\rho \cos^2\phi \, a_z \cdot (-\rho d\rho d\phi) \Big|_{z=-2}$$

$$\Psi_b = +2 \int_{\rho=0}^{1} \rho^2 d\rho \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi = 2\left(\frac{1}{3}\right)\left(\frac{\cancel{2}\pi}{\cancel{2}} + 0\right) = \frac{2}{3}\pi$$

$$\therefore \Psi_t = \frac{2\pi}{3}$$

$$\therefore Q = \Psi = \Psi_s + \Psi_t + \Psi_b = 0 + \frac{2}{3}\pi + \frac{2}{3}\pi = \frac{4\pi}{3} C$$

Problem 4.5 The finite sheet $0 \le x \le 1$, $0 \le y \le 1$ on the $z=0$ plane has a charge density $\rho_s = xy(x^2+y^2+25)^{3/2}$ nc/m^2 find

(a) the total charge on the sheet (b) The electric field at $(0,0,5)$.

(c) the force experienced by a -1 mc charge located at $(0,0,5)$.

Sol (a)

$$Q = \int \rho_s \, ds = \int_0^1\int_0^1 xy(x^2+y^2+25)^{3/2} \, dx\, dy \quad nc.$$

Let $x^2 = u \Rightarrow \frac{du}{dx} = 2x \Rightarrow x\,dx = \frac{1}{2}du = \frac{1}{2}d(x^2)$.

$y^2 = u \Rightarrow \frac{du}{dy} = 2y \Rightarrow y\,dy = \frac{1}{2}du = \frac{1}{2}d(y2)$.

Now integrate w.r.to x^2 & $y2$.

$$\therefore Q = \int_0^1 y \int_0^1 \frac{1}{2}(x^2+y^2+25)^{3/2} \, d(x^2)\, dy \quad nc$$

$$= \frac{1}{2}\int_0^1 y \left[\frac{(x^2+y^2+25)^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_0^1 dy$$

$$= \frac{1}{2}\times\frac{2}{5}\int_0^1 \left[(26+y^2)^{5/2} - (25+y^2)^{5/2}\right]\frac{1}{2}d(y2)$$

$$= \frac{1}{5}\times\frac{1}{2}\left[\frac{(26+y2)^{\frac{5}{2}+1}}{\frac{5}{2}+1} - \frac{(25+y2)^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right]_0^1$$

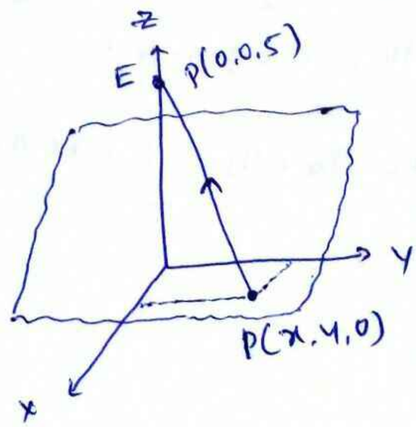
$$= \frac{1}{10}\times\frac{2}{7}\left[\left\{(27)^{7/2} - (26)^{7/2}\right\} - \left\{(26)^{7/2} - (25)^{7/2}\right\}\right]$$

$$= \frac{1}{35}\left[(27)^{7/2} + (25)^{7/2} - 2(26)^{7/2}\right]$$

$$\boxed{Q = 33.15 \text{ nc}}$$

(b)

$$E = \int \frac{\rho_s ds}{4\pi\epsilon_0 R^2} a_R$$



$R = (0-x)a_x + (0-y)a_y + (5-0)a_z$

$R = -x a_x - y a_y + 5 a_z$

$|R| = \sqrt{x^2+y^2+25}$

$$E = \int \frac{\rho_s\, dS\, R}{4\pi\epsilon_0 |R|^3} = \int_0^1\int_0^1 \frac{xy(x^2+y^2+25)^{3/2} \times 10^{-9}(-x a_x - y a_y + 5a_z)\, dx\, dy}{4\pi \times \frac{1}{36\pi \times 10^9}(x^2+y^2+25)^{3/2}}$$

$$E = 9\times 10^9 \times 10^{-9}\left[-\int_0^1 x^2 dx \int_0^1 y\, dy\, a_x - \int_0^1 x\, dx \int_0^1 y^2 dy\, a_y + \int_0^1 5x\, dx \int_0^1 y\, dy\, a_z\right]$$

$$E = 9\left[-\left[\frac{x^3}{3}\right]_0^1\left[\frac{y^2}{2}\right]_0^1 a_x - \left[\frac{x^2}{2}\right]_0^1\left[\frac{y^3}{3}\right]_0^1 a_y + 5\left[\frac{x^2}{2}\right]_0^1\left[\frac{y^2}{2}\right]_0^1 a_z\right]$$

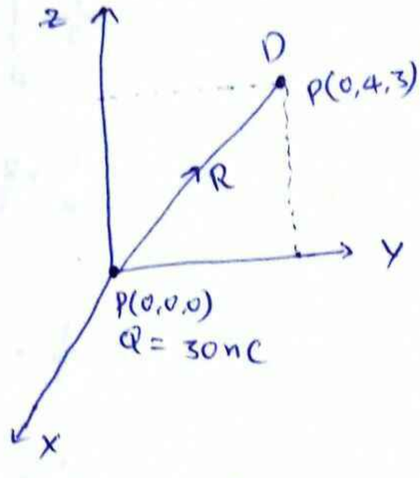
$$E = 9\left[-\frac{1}{3}\cdot\frac{1}{2}a_x - \frac{1}{2}\cdot\frac{1}{3}a_y + 5\cdot\frac{1}{2}\cdot\frac{1}{2}a_z\right] = 9\left[-\frac{1}{6}a_x - \frac{1}{6}a_y + \frac{5}{4}a_z\right]$$

$$\boxed{E = -1.5a_x - 1.5a_y + 11.25a_z \text{ V/m (or) N/C}}$$

(c) $F = EQ = (-1\times 10^{-3})(-1.5a_x - 1.5a_y + 11.25a_z)$

$$\boxed{F = 1.5a_x + 1.5a_y - 11.25a_z} \text{ mN.}$$

Problem (PE 4.7) A point charge of 30 nC is located at the origin while plane $y=3$ carries charge 10 nC/m². Find D at (0,4,3)



Sol

$R = (0-0)a_x + (4-0)a_y + (3-0)a_z = 4a_y + 3a_z$

$|R| = \sqrt{16+9} = 5$

$$D_Q = \frac{Q\, R}{4\pi |R|^3} = \frac{30\times 10^{-9}(4a_y + 3a_z)}{4\pi \times 125}$$

$D_Q = 0.0763a_y + 0.0573a_z$ nC/m²

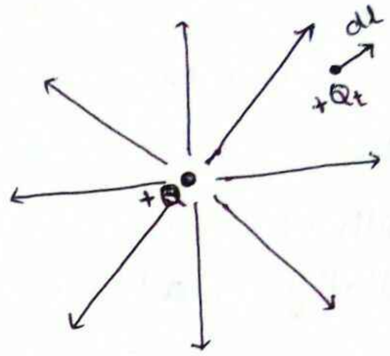
→ plane $y=3$ means sheet is in the xz plane. Hence ρ_s is along a_y direction

$$\therefore D_S = \frac{\rho_s}{2}a_y = \frac{10\times 10^{-9}}{2}a_y = 5\times 10^{-9}a_y$$

$D_S = 5a_y$ nC/m²

$$\boxed{\therefore D = D_Q + D_S = 5.076a_y + 0.0573a_z \text{ nC/m}^2}$$

Work Done :



Consider a Positive charge Q and its electric field E. If a Positive test charge Q_t is placed in this field, it will move due to force of repulsion. Let the movement of the charge Q_t is 'dl'.

→ To keep the charge in equilibrium, it is necessary to apply the force which is equal and opposit to the force exerted by the field in the direction dl

→ Thus keeping the charge in equilibrium means we are moving a charge Q_t through the distance dl in opposit direction to that of field E. Hence the work is done.

From coulomb's law, the force on Q_t is $F = EQ_t$. So that the work done in displacing the charge by dl is

$$dW = -F \cdot dl = -E Q_t \cdot dl$$

The negative sign indicates that the work is being done by an external agent. Thus the total work done is

$$\boxed{W = -Q_t \int E \cdot dl}$$

Electric Potential :

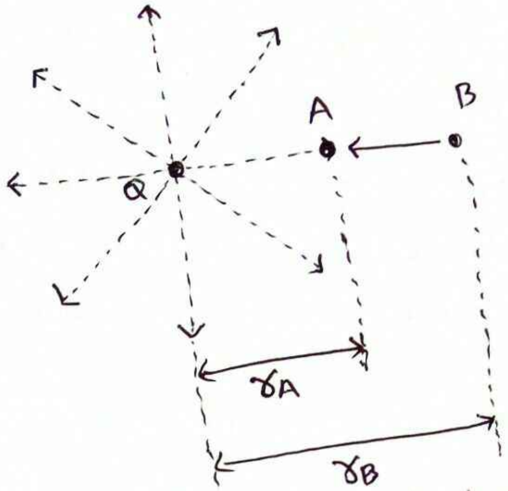
The work done in moving a unit charge from one point to another point in an electric field E is called electric potential (or) Potential difference (or) Electric scalar Potential (or) Potential.

→ It is denoted by V (or) V_{AB} and measured in volts. It is a scalar Quantity.

$$\therefore \quad V = \frac{W}{Q} = -\int E \cdot dl \qquad \boxed{\therefore V = -\int E \cdot dl}$$

Here Q is selected as unit test charge Q_t from work done.

Electric Potential Due to Point Charge :



Consider a point charge Q, located at the origin of a spherical coordinate system, producing E radially in all the directions as shown in the figure.

Assuming free space, the field E due to a point charge Q at a point having radial distance r from the origin is given by $E = \frac{Q}{4\pi\epsilon_0 r^2} a_r$

$V = -\int E \cdot dl = -\int \frac{Q}{4\pi\epsilon_0 r^2} a_r \cdot dr\, a_r$

$V = \frac{-Q}{4\pi\epsilon_0} \int \frac{1}{r^2} dr = \frac{-Q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)$

$V = \frac{Q}{4\pi\epsilon_0 r}$

→ Consider a unit charge which is placed at point B which is at a radial distance of r_B from the origin. It is moved against the direction of E from point B to point A. Point A is at a radial distance of r_A from the origin.

→ The differential length in spherical system is

$$dl = dr\, a_r + r\, d\theta\, a_\theta + r \sin\theta\, d\phi\, a_\phi$$

Hence the potential difference V_{AB} b/w points A & B is given by

$$V_{AB} = -\int_B^A E \cdot dl$$

But $B \Rightarrow r_B$ &

$A \Rightarrow r_A$

$$\therefore V_{AB} = -\int_{r_B}^{r_A} \left(\frac{Q}{4\pi\epsilon_0 r^2} a_r\right) \cdot (dr\, a_r) = -\int_{r_B}^{r_A} \frac{Q}{4\pi\epsilon_0 r^2} dr$$

$$\therefore V_{AB} = \frac{-Q}{4\pi\epsilon_0} \int_{r_B}^{r_A} r^{-2} dr = \frac{-Q}{4\pi\epsilon_0} \left[-\frac{1}{r}\right]_{r_B}^{r_A} = \frac{-Q}{4\pi\epsilon_0}\left[\frac{-1}{r_A} + \frac{1}{r_B}\right]$$

$$\boxed{\therefore V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] V}$$

When $r_B > r_A$, $\frac{1}{r_B} < \frac{1}{r_A}$ and V_{AB} is positive. This indicates that the work is done by external source in moving unit charge from B to A.

Absolute Potential :

The absolute Potential at any Point in an electric field is defined as the work done in moving a unit test charge from infinity to the Point, against the direction of the field.

→ Consider Potential difference V_{AB} due to movement of unit charge from B to A in a field of Point charge Q is

$$V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Now charge is moved from infinity to Point A i.e $r_B = \infty \Rightarrow \frac{1}{r_B} = \frac{1}{\infty} = 0$

$$\therefore V_{AB} = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - 0\right] = \frac{Q}{4\pi\epsilon_0 r_A} \text{ V.}$$

It is also called Potential of Point A denoted by V_A

$$\boxed{\therefore V_A = \frac{Q}{4\pi\epsilon_0 r_A} \text{ V.}}$$

This is also called absolute Potential of Point A.

Similarly absolute Potential of Point B can be defined as

$$\boxed{\therefore V_B = \frac{Q}{4\pi\epsilon_0 r_B} \text{ V}}$$

This is the work done in moving unit charge from infinity at Point B.

→ Hence the Potential difference can be expressed as the difference between the absolute Potentials of the two points.

$$\therefore V_{AB} = V_A - V_B$$

→ There fore the Potential (or) absolute Potential at any Point which is at a distance r from the origin of a spherical System, where Point charge Q is located. is given by

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}} \longrightarrow (1)$$

→ If the point charge Q in eq (1) is not located at the origin but at point whose position vector r', the potential $V(x,y,z)$ at r becomes

$$\boxed{V(x,y,z) = V(r) = \frac{Q}{4\pi\epsilon_0 |r - r'|}}$$

→ For n point charges $Q_1, Q_2, \dots Q_n$ located at points with position vectors $r_1, r_2 \dots r_n$, the potential at r is

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r - r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r - r_2|} + \dots + \frac{Q_n}{4\pi\epsilon_0 |r - r_n|}$$

$$\boxed{V(r) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{Q_k}{|r - r_k|}} \longrightarrow (2) \quad \text{(Point charge)}$$

→ For continuous charge distributions, we replace Q_k in eq (2) with charge element $\rho_l\,dl$, $\rho_s\,ds$, $\rho_v\,dv$ and the summation becomes an integration, so the potential at r becomes

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_l(r')\,dl'}{|r - r'|} \quad \text{(line charge)}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\rho_s(r')\,ds'}{|r - r'|} \quad \text{(Surface charge)}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_v \frac{\rho_v(r')\,dv'}{|r - r'|} \quad \text{(volume charge)}$$

Problem

Two point charges $-4\,\mu C$ and $5\,\mu C$ are located at $(2,-1,3)$ and $(0,4,-2)$, respectively. Find the potential at $(1,0,1)$, assuming zero potential at infinity.

Sol

Given $Q_1 = -4\,\mu C$, $Q_2 = 5\,\mu C$

Potential $V(r) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q_1}{|r-r_1|} + \frac{Q_2}{|r-r_2|}\right]$

$$|r - r_1| = |(1,0,1) - (2,-1,3)| = |-1,1,-2| = \sqrt{1+1+4} = \sqrt{6}$$

$$|r - r_2| = |(1,0,1) - (0,4,-2)| = |1,-4,3| = \sqrt{1+16+9} = \sqrt{26}$$

$$\therefore V(1,0,1) = \frac{10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right]$$

$$= 9 \times 10^{3}\,(-1.633 + 0.9806)$$

$$= -5.872\ \text{kV}$$

W.Hayt

Given the Electric flux density $D = 0.3r\, a_r\ nc/m^2$ in freespace.

(a) Find electric field E at a point $P(r=2, \theta = 25^\circ, \phi = 90^\circ)$

(b) Find the total charge with in radius of sphere $r=3$

(c) Find the total electric flux leaving the surface of sphere $r=4$.

Sol (a) Given $D = 0.3r\ a_r\ nc/m^2$

$$D = \epsilon_0 E \Rightarrow E = \frac{D}{\epsilon_0} = 36\pi \times 10^9 \times 0.3r \times 10^{-9}\ a_r$$

$$E = 67.8\ a_r\quad V/m$$

If $D = 0.3r^2 a_r\ nc/m^2$

then (a) $135.5\ a_r$ V/m

(b) 305 nc

(c) 965 nc

(b) $Q(r=3)$

$$Q = \oint_S D \cdot dS = \oint_S (0.3r)\, a_r \cdot (r^2 \sin\theta\, d\theta\, d\phi)\, a_r$$

$$Q = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} (0.3r^3) \sin\theta\, d\theta\, d\phi \Big|_{r=3}$$

$$Q = (0.3)(3)^3\, 2\pi\, (-\cos\theta)_0^{\pi} = (0.3)(27)(2\pi)(2)$$

$$Q = 101.78\ nC.$$

W.K.T $D = \frac{Q}{4\pi r^2} \Rightarrow Q = D 4\pi r^2$

(c) $\psi = Q\ (r=4)$

$$\psi = D(4\pi r^2) = 0.3r\,(4\pi r^2) = (0.3)(4\pi)(r^3)$$

$$\psi = (0.3)(4\pi)(4)^3 = 241.27\ nC.$$

Ex.4.12 (140)

Given the Potential $V = \frac{10}{r^2} \sin\theta \cos\phi$,

(a) find the electric flux density D at $(2, \pi/2, 0)$.

(b) Calculate the work done in moving a $10\,\mu C$ charge from point $A(1, 30^\circ, 120^\circ)$ to $B(4, 90^\circ, 60^\circ)$.

Sol

(a) $D = \epsilon_0 E$

$$E = -\nabla V = -\left[\frac{dV}{dr} a_r + \frac{1}{r}\frac{dV}{d\theta} a_\theta + \frac{1}{r\sin\theta}\frac{dV}{d\phi} a_\phi\right]$$

$$E = -\left[10 \sin\theta \cos\phi \frac{d}{dr} r^{-2} a_r + \frac{1}{r}\frac{10}{r^2}\cos\phi \frac{d}{d\theta}\sin\theta\, a_\theta + \frac{1}{r\sin\theta}\frac{10}{r^2}\sin\theta \frac{d}{d\phi}\cos\phi\, a_\phi\right]$$

$$E = -\left[10 \sin\theta \cos\phi \frac{(-2)}{r^3} a_r + \frac{10}{r^3}\cos\phi \cos\theta\, a_\theta + \frac{10}{r^3}(-\sin\phi)\, a_\phi\right]$$

$$E = \frac{20}{r^3}\sin\theta\cos\phi\, a_r - \frac{10}{r^3}\cos\theta\cos\phi\, a_\theta + \frac{10}{r^3}\sin\phi\, a_\phi$$

at $(2, \pi/2, 0)$ $\quad D = \epsilon_0 E(r=2, \theta=\pi/2, \phi=0)$

$$D = \epsilon_0\left[\frac{20}{8}(1)(1)\, a_r - 0\, a_\theta + 0\, a_\phi\right]$$

$$D = \epsilon_0 (2.5)\, a_r \ \ C/m^2$$

$$\boxed{D = 22.1\, a_r \ pC/m^2}$$

(b) $W = -Q\int_A^B E\cdot dl = Q V_{AB} = Q(V_B - V_A)$

$$W = 10\left[\frac{10}{16}\sin 90^\circ \cos 60^\circ - \frac{10}{1}\sin 30^\circ \cos 120^\circ\right]\cdot 10^{-6}$$

$$W = 10\left[\frac{10}{32} - \frac{-5}{2}\right]\cdot 10^{-6} = 28.125\,\mu J \qquad \boxed{\therefore W = 28.125\,\mu J}$$

Relation between E and V (Potential Gradient) :

Electric field intensity(E) is equal to the negative gradient of electric Potential(V).

i.e $$\boxed{E = -\nabla V}$$

Proof :

The electric potential is given by $V = -\int E \cdot dl$

Apply vector differential operator on both sides

$$\nabla V = -\nabla \int E \cdot dl$$

$$-\nabla V = \left(\frac{d}{dx} a_x + \frac{d}{dy} a_y + \frac{d}{dz} a_z\right) \int (E_x a_x + E_y a_y + E_z a_z) \cdot (dx a_x + dy a_y + dz a_z)$$

$$-\nabla V = \left(\frac{d}{dx} a_x + \frac{d}{dy} a_y + \frac{d}{dz} a_z\right) \int (E_x dx + E_y dy + E_z dz)$$

$$-\nabla V = \frac{d}{dx} \int E_x dx\, a_x + \frac{d}{dy} \int E_y dy\, a_y + \frac{d}{dz} \int E_z dz\, a_z$$

$$-\nabla V = E_x a_x + E_y a_y + E_z a_z$$ [∵ Integration & Differentiation are cancelled]

$$-\nabla V = E$$

$$\boxed{\therefore\ E = -\nabla V}$$

(or)

The Electric field intensity $E = \frac{Q}{4\pi\epsilon_0 r^2} a_r \longrightarrow (1)$

we know that $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2} a_r \Rightarrow \frac{1}{r^2} a_r = -\nabla\left(\frac{1}{r}\right) \longrightarrow (2)$

Substitute (2) in (1) $E = \frac{Q}{4\pi\epsilon_0}\left[-\nabla\left(\frac{1}{r}\right)\right] = -\nabla \frac{Q}{4\pi\epsilon_0 r} = -\nabla V$ $\quad \Big| \because V = \frac{Q}{4\pi\epsilon_0 r}$

$\boxed{\therefore\ E = -\nabla V}$ Electric field is negative potential gradient.

* Potential is scalar. Gradient of scalar function is a vector function.

Maxwell's Two equations for Electro Static field :

→ Gauss's law in Point form is called Maxwell's first equation.

Equation 1 : The divergence of the electric flux density (D) is equal to the volume charge density (ρ_v).

i.e., $\boxed{\nabla \cdot D = \rho_v}$

Proof : from the definition of Gauss's law

$$Q = \Psi = \oint_S D \cdot dS$$

Apply divergence theorem to the above equation.

$$Q = \oint_S D \cdot dS = \int_V \nabla \cdot D \, dv \longrightarrow (1)$$

→ Total charge enclosed by the volume is

$$Q = \int_V \rho_v \, dv \longrightarrow (2)$$

equating eq (1) & eq (2)

$$\int_V \nabla \cdot D \, dv = \int_V \rho_v \, dv$$

$$\boxed{\nabla \cdot D = \rho_v}$$

⇒ Conservative field (or) irrotational field is called Maxwell's second equation.

Equation 2 : In conservative field the curl of electric field intensity is equal to zero

i.e., $\boxed{\nabla \times E = 0}$

Proof :

→ The line integral of a vector field **E** around a closed path is zero. Such field is called conservative field.

→ Physically no work is done in moving a charge along a closed path in an electrostatic field.

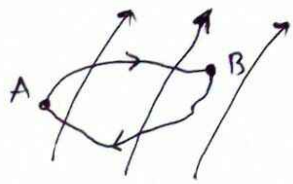


fig: Conservative nature of electrostatic field.

→ Therefore with in the closed loop, potential is zero. i.e., $V_{AB} + V_{BA} = 0$

$$\Rightarrow V = 0$$

$$\therefore -\oint E \cdot dl = 0 \Rightarrow \oint E \cdot dl = 0$$

Apply Stokes theorem to the above equation.

$$\oint_L E \cdot dl = \int_S (\nabla \times E) \cdot ds = 0$$

$$\boxed{\therefore \nabla \times E = 0}$$

Poisson's and Laplace's Equations :

Poisson's Equation :

$$\boxed{\nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho_v}{\epsilon_0}}$$

Proof :

from Maxwell's first equation

$$\nabla \cdot D = \rho_v$$

$$\nabla \cdot \epsilon_0 E = \rho_v$$

$$\epsilon_0 \nabla \cdot E = \rho_v$$

$$\epsilon_0 \nabla \cdot (-\nabla V) = \rho_v$$

$$-\nabla^2 V = \rho_v / \epsilon_0$$

$\because D = \epsilon_0 E$

$\because E = -\nabla V$

$$\boxed{\therefore \nabla \cdot \nabla V = \nabla^2 V = \frac{-\rho_v}{\epsilon_0}}$$

This equation is called Poisson's equation

Laplace's Equation :

$$\boxed{\nabla^2 V = 0}$$

Proof :

Consider a charge free region (insulator) . The value of ρ is zero since there are no free charges in dielectrics.

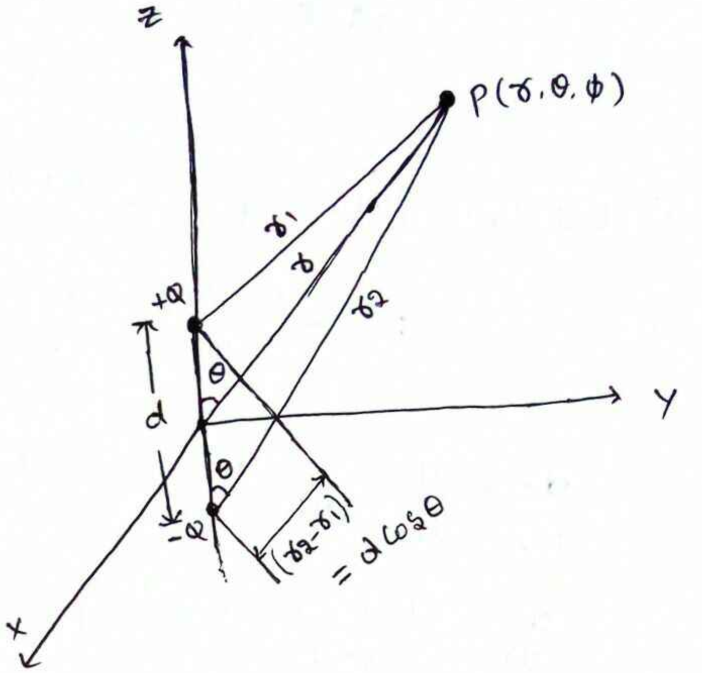
Substitute $\rho=0$ in Poisson's equation .

$$\nabla^2 V = \frac{-0}{\epsilon_0} = 0$$

$\boxed{\therefore \nabla^2 V = 0}$ This equation is called Laplace's equation.

Dipole and Dipole Moment :

Electric Dipole :



→ An electric dipole is formed when two point charges of equal magnitude but opposit sign are seperated by a small distance.

→ Consider a point $P(r, \theta, \phi)$ in a spherical coordinate system. It is required to find E due to an electric dipole at point P.

→ The distance of seperation of charges i.e., 'd' is very small compared to the distances r_1, r_2 and r.

* In spherical coordinates, the potential at point P due to the charge +Q is given by $V_1 = \frac{+Q}{4\pi\epsilon_0 r_1}$

* The potential at point P due to the charge −Q is given by

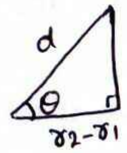
$$V_2 = \frac{-Q}{4\pi\epsilon_0 r_2}$$

* The total potential at point P is the algebraic sum of V_1 and V_2

$$\therefore V_1 + V_2 = V = \frac{Q}{4\pi\epsilon_0 r_1} - \frac{Q}{4\pi\epsilon_0 r_2}$$

$$\therefore V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$$

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right]$$



$\cos\theta = \frac{r_2 - r_1}{d} \Rightarrow r_2 - r_1 = d\cos\theta$

→ where r_1 & r_2 are the distances between P & +Q and P & −Q respectively.

If $r >> d$, $r_2 - r_1 \simeq d\cos\theta$, $r_1 r_2 \simeq r^2$. then above equation becomes

$$\boxed{V = \frac{Q}{4\pi\epsilon_0}\left[\frac{d\cos\theta}{r^2}\right]}$$

This is the Potential

$\therefore$ Electric field $E = -\nabla V$

$$E = -\left[\frac{\partial V}{\partial r} a_r + \frac{1}{r}\frac{\partial V}{\partial \theta} a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} a_\phi\right] \longrightarrow (1)$$

$$\left.\begin{aligned}
\frac{\partial V}{\partial r} &= \frac{\partial}{\partial r}\left[\frac{Q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}\right] = \frac{Q\,d\cos\theta}{4\pi\epsilon_0}\frac{\partial}{\partial r}(r^{-2}) \\
\frac{\partial V}{\partial r} &= \frac{Q\,d\cos\theta}{4\pi\epsilon_0}\left(-2.r^{-2-1}\right) = -\frac{Q\,d\cos\theta}{4\pi\epsilon_0}\left(\frac{2}{r^3}\right) = \frac{-2Q\,d\cos\theta}{4\pi\epsilon_0 r^3} \\
\frac{\partial V}{\partial \theta} &= \frac{\partial}{\partial \theta}\left[\frac{Q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}\right] = \frac{Qd}{4\pi\epsilon_0 r^2}(-\sin\theta) = \frac{-Q\,d\sin\theta}{4\pi\epsilon_0 r^2} \\
\frac{\partial V}{\partial \phi} &= 0
\end{aligned}\right\} \longrightarrow (2)$$

Substitute (2) in (1)

$$E = -\left[\frac{-2Q\,d\cos\theta}{2\,4\pi\epsilon_0 r^3} a_r \times \frac{2}{2} + \frac{1}{r}\left(\frac{-Q\,d\sin\theta}{4\pi\epsilon_0 r^2}\right) a_\theta + 0\right]$$

$$\boxed{E = \frac{Qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\, a_r + \sin\theta\, a_\theta\right]}$$

Dipole moment: It is the product of charge and the seperation between charges. It is a vector quantity & has the direction along distance vector d i.e. $\boxed{\vec{P} = Q\vec{d}}$

Since $d\cos\theta = d\,a_r\cos\theta = d\cdot a_r$

* then dipole moment can be written as $V = \frac{Q}{4\pi\epsilon_0}\left(\frac{d\cdot a_r}{r^2}\right) = \frac{Qd\cdot a_r}{4\pi\epsilon_0 r^2}$

$$\boxed{\therefore V = \frac{P\cdot r}{4\pi\epsilon_0 |r|^3}} \quad \therefore a_r = \frac{r}{|r|}$$

* If the dipole centre is not at the origin but at r', then dipole moment can be written as $\boxed{V(r) = \frac{P\cdot(r-r')}{4\pi\epsilon_0 |r-r'|^3}}$

Energy Density in Electrostatic Fields:

→ To determine the energy present in an assembly of charges, we must ~~first~~ determine the amount of work necessary to assemble them.

→ Suppose we wish to position ~~Consider~~ three point charges Q_1, Q_2 and Q_3 ~~are~~ in an initially ~~in~~ empty space shown in fig.

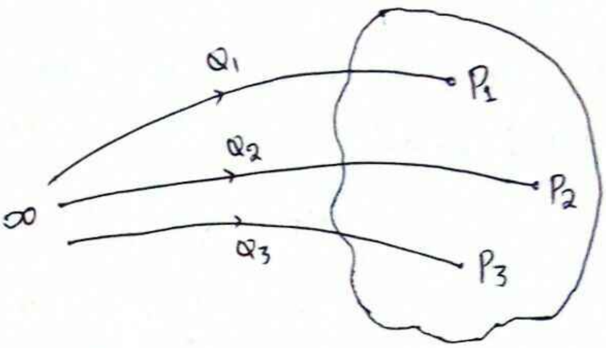


fig: Assembling of charges

→ No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field $[W=0]$.

→ The work done in transferring Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1. [i.e, Q_2V_{21}].

→ ~~The~~ Similarly, the work done in transferring Q_3 from infinity to P_3 is equal to the $Q_3(V_{32}+V_{31})$. where V_{32} & V_{31} are the potentials at P_3 due to Q_2 & Q_1 respectively.

* Hence the total work done in positioning the three charges is

$$W_E = W_1 + W_2 + W_3$$
$$= 0 + Q_2V_{21} + Q_3(V_{31}+V_{32}) \longrightarrow (1)$$

* If the charges were positioned in reverse order,

$$W_E = W_3 + W_2 + W_1$$
$$= 0 + Q_2V_{23} + Q_1(V_{12}+V_{13}) \longrightarrow (2)$$

→ where V_{23} is the potential at P_2 due to Q_3, V_{12} & V_{13} are the potential at P_1 due to Q_2 & Q_3. Adding eq (1) & (2)

$$2W_E = Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) + Q_3(V_{31}+V_{32})$$
$$2W_E = Q_1V_1 + Q_2V_2 + Q_3V_3$$
$$\boxed{W_E = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3)}$$

where V_1, V_2 & V_3 are total potentials at P_1, P_2 & P_3 respectively.

→ If there are n- Point charges, the above equation becomes

$$\boxed{W_E = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k} \quad \text{Joules.}$$

→ If, instead of point charges, the region has a continuous charge distribution, the summation becomes integration i.e.,

$$W_E = \frac{1}{2}\int \rho_L V\, dl \rightarrow \text{line charge}$$

$$W_E = \frac{1}{2}\int \rho_S V\, dS \rightarrow \text{Surface charge}$$

$$W_E = \frac{1}{2}\int \rho_v V\, dv \rightarrow \text{Volume charge}$$

→ Since $\rho_v = \nabla \cdot D$ then volume charge equation becomes

$$W_E = \frac{1}{2}\int_v (\nabla \cdot D)\, V\, dv$$

→ But for any vector A & scalar V, the identity $\boxed{(\nabla \cdot A)V = \nabla \cdot VA - A \cdot \nabla V}$

Apply this identity to the above equation.

$$W_E = \frac{1}{2}\int_v (\nabla \cdot VD)\, dv - \frac{1}{2}\int_v (D \cdot \nabla V)\, dv$$

Apply divergence theorem to the 1st term on the right-hand side of eq.

$$W_E = \frac{1}{2}\oint_S (VD) \cdot dS - \frac{1}{2}\int_v (D \cdot \nabla V)\, dv$$

The work done for a closed loop is zero.

$$W_E = 0 - \frac{1}{2}\int_v (D \cdot \nabla V)\, dv$$

$$W_E = \frac{1}{2}\int_v (D \cdot E)\, dv = \frac{1}{2}\int_v (\epsilon_0 E \cdot E)\, dv \qquad \because E = -\nabla V, \; D = \epsilon_0 E$$

$$\boxed{\therefore W_E = \frac{1}{2}\int_v (D \cdot E)\, dv = \frac{1}{2}\int \epsilon_0 E^2\, dv = \frac{1}{2}\int_v \frac{D^2}{\epsilon_0}\, dv}$$

→ from this, we can define electrostatic energy density w_E (in J/m^3) as

$$\boxed{w_E = \frac{dW_E}{dv} = \frac{1}{2} D \cdot E = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\frac{D^2}{\epsilon_0}}$$

$$\Rightarrow W_E = \int_v w_E\, dv.$$

EF-17.1

Problem 4.13

Two dipoles with dipole moments $-5a_z$ nc/m and $9a_z$ nc/m are located at Points $(0,0,-2)$ & $(0,0,3)$ respectively. Find potential at origin.

Sol

Dipole moment for two point $V = \frac{1}{4\pi\epsilon_0}\left[\frac{p_1 \cdot r_1}{|r_1|^3} + \frac{p_2 r_2}{|r_2|^3}\right]$

$\therefore p_1 = -5a_z$ & $r_1 = (0,0,0)-(0,0,-2) = 2a_z, \quad r_1 = |r_1| = \sqrt{2^2} = 2.$

$p_2 = 9a_z$ & $r_2 = (0,0,0)-(0,0,3) = -3a_z, \quad r_2 = |r_2| = \sqrt{3^2} = 3.$

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{-5\cdot(2)}{|2|^3} + \frac{9(-3)}{|3|^3}\right] \times 10^{-9}$$

$$V = 9\times10^9\left[\frac{-10}{8} - \frac{27}{27}\right] \times 10^{-9}$$

$$V = 9\left(\frac{-10}{8} - 1\right) = 9\left(\frac{-10-8}{8}\right) = 9\left(\frac{-18}{8}\right) = -20.25\,V$$

$$\boxed{\therefore V = -20.25}$$

Problem (3.19)

The point charges -1nC, 4nC and 3nC are located at $(0,0,0)$, $(0,0,1)$ and $(1,0,0)$ respectively. Find the energy in the system

Sol

$Q_1 = -1$nc at $(0,0,0)$

$Q_2 = 4$nc at $(0,0,1)$

$Q_3 = 3$nc at $(1,0,0)$

$$\therefore W = W_1 + W_2 + W_3 = 0 + Q_2 V_{21} + Q_3(V_{31} + V_{32})$$

$$W = Q_2 \cdot \left\{\frac{Q_1}{4\pi\epsilon_0 |(0,0,1)-(0,0,0)|}\right\} + \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1}{|(1,0,0)-(0,0,0)|} + \frac{Q_2}{|(1,0,0)-(0,0,1)|}\right]$$

$$W = Q_2\left(\frac{Q_1}{4\pi\epsilon_0(\sqrt{1^2})}\right) + \frac{Q_3}{4\pi\epsilon_0}\left[\frac{Q_1}{\sqrt{1^2}} + \frac{Q_2}{\sqrt{1^2+1^2}}\right]$$

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1 Q_2}{1} + \frac{Q_1 Q_3}{1} + \frac{Q_2 Q_3}{\sqrt{2}}\right)$$

$$W = 9\times10^9\left[(-1\times4) + (-1\times3) + \frac{(4\times3)}{\sqrt{2}}\right] \times 10^{-18}$$

$$W = 9\times10^{-9}\left[\frac{12}{\sqrt{2}} - 7\right] = 13.37\text{ nJ}$$

$$W = \frac{1}{2}\sum_{k=1}^{3} Q_k V_k = \frac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3)$$

$$W = \frac{1}{2}\left[Q_1(V_{12}+V_{13}) + Q_2(V_{21}+V_{23}) + Q_3(V_{31}+V_{32})\right]$$

$$W = \frac{Q_1}{2}\left[\frac{Q_2}{4\pi\epsilon_0(1)} + \frac{Q_3}{4\pi\epsilon_0(1)}\right] + \frac{Q_2}{2}\left[\frac{Q_1}{4\pi\epsilon_0(1)} + \frac{Q_3}{4\pi\epsilon_0(\sqrt{2})}\right] + \frac{Q_3}{2}\left\{\frac{Q_1}{4\pi\epsilon_0(1)} + \frac{Q_2}{4\pi\epsilon_0(\sqrt{2})}\right\}$$

$$W = \frac{1}{4\pi\epsilon_0}\left[Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right] = \frac{1}{4\pi\epsilon_0}\left[-4-3+\frac{12}{\sqrt{2}}\right]\times 10^{-18}$$

$$W = 9\times 10^{9}\times 10^{-18}\left(\frac{12}{\sqrt{2}} - 7\right) = 13.37\ nJ$$

Problem 3.1a) Point charges $Q_1 = 1nC$, $Q_2 = -2nC$, $Q_3 = 3nC$ and $Q_4 = -4nC$ are positioned one at a time and in that order at $(0,0,0)$, $(1,0,0)$, $(0,0,-1)$ and $(0,0,1)$ respectively. Calculate the energy in the system after each charge is positioned.

(Ans: 0, −18 nJ, −29.18 nJ, −68.27 nJ).

Capacitance: * YVNR * EF-18

Capacitor have two conductors carying equal but opposit charges. The conductors are sometimes referred to as the plates of the capacitor. The plates may be seperated by free spaces or a dielectric.

→ Consider two conductor capacitor as shown in fig. The conductors are maintained at a potential difference V given by

$$V = V_2 - V_2 = -\int_2^1 E \cdot dl$$

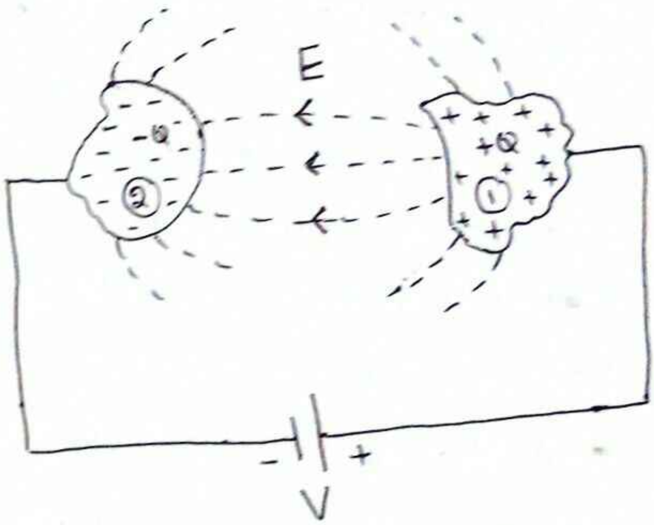


fig. Two conductor capacitor

where E = Electric field existing between the conductors

Conductor 1 is assumed to carry a positive charge.

Def: Capacitance of a capacitor can be defined as the ratio of the magnitude of the charge on one of the plate to the potential difference between them. i.e.,

$$C = \frac{Q}{V} = \frac{\oint D \cdot ds}{-\int E \cdot dl}$$

$$= \frac{-\oint \epsilon_0 E \cdot ds}{\int E \cdot dl}$$

$$\boxed{\therefore C = \epsilon \left| \frac{\oint E \cdot ds}{\int E \cdot dl} \right|} \quad (or) \quad \boxed{C = \frac{\epsilon \oint E \cdot ds}{\int E \cdot dl}}$$

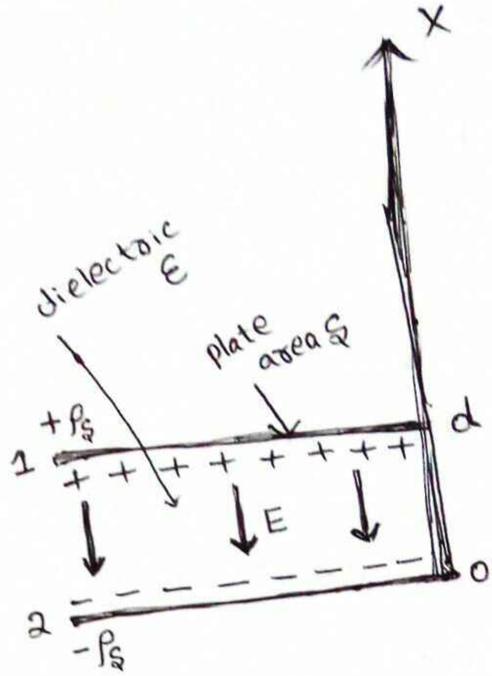
* The negative sign befor $V = -\int E \cdot dl$ has been dropped because we are interested in the absolute value of V.

→ Capacitor is denoted with C and is measured in farads (F).

→ There are three types of capacitors.

1. Parallel-plate capacitor
2. Coaxial capacitor
3. Spherical capacitor.

1. Parallel-plate capacitor :



Consider a parallel-plate capacitor is shown in fig. Suppose that each of the plates has an area S and they are seperated by a distance 'd'.

→ The space between the plates is filled with a dielectric of permitivity ε.

→ The plate 1 carries the positive charge and is distributed over its surface with a charge density $+\rho_S$.

→ The plate 2 carries the negative charge and is distributed over its surface with a charge density $-\rho_S$.

$\therefore \rho_S = \frac{dQ}{dS} = \frac{Q}{S}$

The surface charge $\rho_S = \frac{Q}{S}$ ⟶ (1)

→ To find the potential difference, let us obtain E between the plates.

Assuming plate 1 to be infinite sheet of charge,

$$E_1 = \frac{\rho_S}{2\epsilon} a_n = \frac{\rho_S}{2\epsilon}(-a_x) \quad V/m$$

While for plate 2, we can write

$$E_2 = \frac{-\rho_S}{2\epsilon} a_n = \frac{-\rho_S}{2\epsilon} a_x \quad V/m$$

→ The electric field existing between the two plates having equal and opposit charges is given by

$$\therefore \quad E = E_1 + E_2$$

$\therefore E = E_1 + E_2$

$$E = \frac{\rho_s}{2\epsilon}(-a_x) + \frac{-\rho_s}{2\epsilon} a_x$$

$$E = -\frac{\rho_s}{2\epsilon} a_x - \frac{\rho_s}{2\epsilon} a_x$$

$$E = \frac{-\cancel{2}\rho_s}{\cancel{2}\epsilon} a_x$$

$$E = \frac{-\rho_s}{\epsilon} a_x \longrightarrow (2)$$

Substitute eq(1) in (2). then

$$E = \frac{-Q}{\epsilon S} a_x$$

→ The Potential difference is given by

$$V = -\int_2^1 E \cdot dl$$

differential length in cartesian coordinate system $= dx a_x + dy a_y + dz a_z$

$$V = -\int_0^d \left(\frac{-Q}{\epsilon S} a_x\right) \cdot dx\, a_x$$

$$V = \frac{Q}{\epsilon S} \int_0^d 1\, dx$$

$$V = \frac{Q}{\epsilon S} [x]_0^d$$

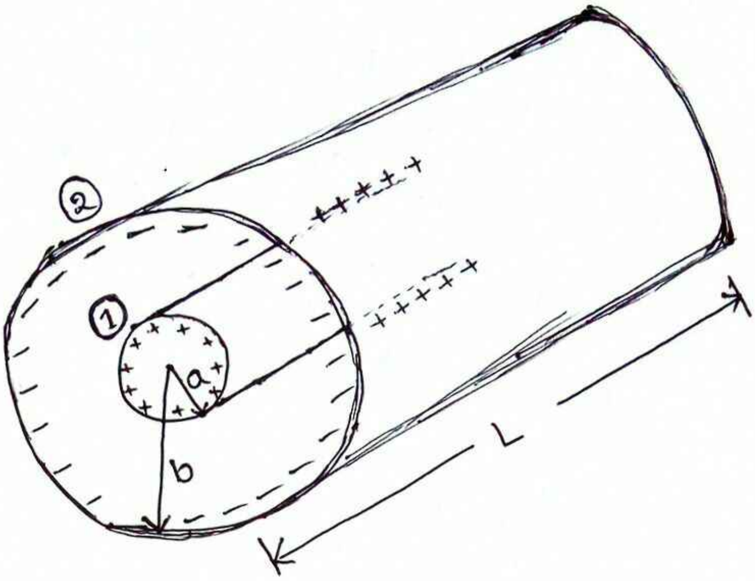
$$V = \frac{Qd}{\epsilon S}$$

→ The capacitance is the ratio of charge Q to voltage V.

$$C = \frac{Q}{V} = \frac{\cancel{Q}}{\cancel{Q}d/\epsilon S} = \frac{\epsilon S}{d}$$

$$\boxed{\therefore C = \frac{\epsilon S}{d}}$$

2. Coaxial Capacitor :



This is essentially a coaxial cable (or) coaxial cylindrical capacitor. Consider a length 'L' of two coaxial conductors of inner radius 'a' and outer radius 'b' (b>a) as shown in fig.

→ let the space between the conductors be filled with a homogeneous dielectric with permitivity ϵ.

→ The inner conductor 1 carries the positive charge and is distributed along the line with a charge density $+\rho_L$

→ The outer conductor 2 carries the negative charge and is distributed along the line with a charge density $-\rho_L$

The line charge $\rho_L = \frac{Q}{L}$

$\rho_L = \frac{dQ}{dl} = \frac{Q}{L}$

→ Assuming cylindrical coordinate system, Electric field (E) will be radial from inner to outer conductor

* Use Gauss's law to obtain Electric field b/w inner & outer conductors.

$$\Psi = Q_{enclosed} = \oint_S D \cdot ds$$

$$Q = \oint_S D\rho a_\rho \cdot \rho d\phi dz a_\rho$$

$$Q = \oint_S D_\rho \rho d\phi dz \quad (1)$$

$$Q = \rho D_\rho \int_{z=0}^{L} dz \int_{\phi=0}^{2\pi} d\phi$$

$$Q = \rho D_\rho [z]_0^L [\phi]_0^{2\pi}$$

In cylindrical system

$D = D_\rho a_\rho$

$ds = \rho d\phi dz a_\rho$

$dl = d\rho a_\rho + \rho d\phi a_\phi + dz a_z$

$$Q = \rho D_\rho L 2\pi$$

$$D_\rho = \frac{Q}{2\pi \rho L}$$

Hence $D = D_\rho a_\rho = \frac{Q}{2\pi\rho L} a_\rho$

$$E = \frac{D}{\epsilon_0}$$

$$E = \frac{Q}{2\pi\epsilon_0 \rho L} a_\rho$$

→ The Potential difference is given by

$\rho \rightarrow$ Gaussian cylindrical surface of radius

$a < \rho < b$

$\int \frac{1}{x} dx = \log x$

$$V = -\int_2^1 E \cdot dl$$

$$= -\int_b^a \frac{Q}{2\pi\epsilon\rho L} a_\rho \cdot d\rho\, a_\rho$$

$$= \frac{-Q}{2\pi\epsilon L} \int_b^a \frac{1}{\rho} d\rho \quad (1)$$

$$= \frac{-Q}{2\pi\epsilon L} [\log \rho]_b^a$$

$$= \frac{-Q}{2\pi\epsilon L} [\log a - \log b]$$

$$= \frac{Q}{2\pi\epsilon L} [\log b - \log a]$$

$$= \frac{Q}{2\pi\epsilon L} \log (b/a)$$

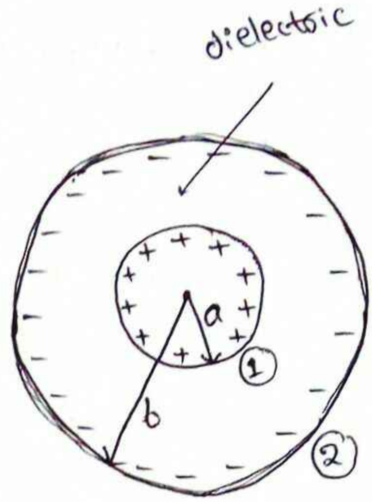
→ The capacitance is the ratio of charge Q to Voltage V

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{2\pi\epsilon L}\log(b/a)} = \frac{2\pi\epsilon L}{\log(b/a)}$$

$$\boxed{\therefore C = \frac{2\pi\epsilon L}{\log(b/a)}}$$

* **Note:** 'log' is replaced by 'ln' in this derivation.

3. Spherical Capacitor :



This is the case of two concentric spherical conductors. Consider the inner sphere of radius 'a' and outer sphere of radius 'b' (b>a) seperated by a dielectric medium with permitivity ϵ as shown in fig.

→ We assume charges +Q and −Q on the inner and outer spheres respectively.

→ Use Gauss's law to obtain Electric field b/w inner & outer sphers.

$$\Psi = Q_{enclosed} = \oint_S D \cdot ds$$

$$Q = \oint D_r a_r \cdot r^2 \sin\theta \, d\theta \, d\phi \, a_r$$

$$Q = D_r r^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta \quad (1)$$

$$Q = D_r r^2 \left[\phi\right]_0^{2\pi} \left[-\cos\theta\right]_0^{\pi}$$

$$Q = D_r r^2 (2\pi) \cdot \left[-(-1-1)\right]$$

$$Q = D_r 4\pi r^2$$

$$D_r = \frac{Q}{4\pi r^2}$$

Hence $D = D_r a_r = \dfrac{Q}{4\pi r^2} a_r$

$$E = \frac{D}{\epsilon_0}$$

$$E = \frac{Q}{4\pi \epsilon r^2} a_r$$

In spherical system

$D = D_r a_r$

$ds = r^2 \sin\theta \, d\theta \, a_r$

$dl = dr a_r + r d\theta a_\theta + r\sin\theta \, d\phi a_\phi$.

→ The Potential difference is given by

$$V = -\int_2^1 E \cdot dl$$

$$V = -\int_b^a \frac{Q}{4\pi\epsilon r^2} a_r \cdot dr\, a_r$$

$$V = -\int_b^a \frac{Q}{4\pi\epsilon r^2} dr \quad (1).$$

$r \rightarrow$ Gaussian spherical Surface of radius $a < r < b$

$$V = -\frac{Q}{4\pi\epsilon} \int_b^a \frac{1}{r^2} dr$$

$$V = \frac{-Q}{4\pi\epsilon} \left[\frac{-1}{r}\right]_b^a$$

$$V = \frac{Q}{4\pi\epsilon} \left[\frac{1}{r}\right]_b^a$$

$$V = \frac{Q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b}\right]$$

→ The capacitance is the ratio of charge Q to voltage V

$$C = \frac{Q}{V}$$

$$C = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$$\boxed{\therefore C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}}$$

Problem (6.13) A cylindrical capacitor has radii $a = 1cm$ & $b = 2.5cm$. If the space b/w the plates is filled with an inhomogeneous dielectric with $\epsilon_r = (10+\rho)/\rho$, where ρ in centimeters. find the capacitance per meter of the capacitor.

Sol

Electric field due to line charge $E = \frac{\rho_L}{2\pi\epsilon\rho} a_\rho = \frac{Q}{2\pi\epsilon\rho L} a_\rho$ $\quad \left[\because \rho_L = \frac{Q}{L}\right]$

$$\therefore V = -\int_b^a \frac{Q}{2\pi\epsilon_0\epsilon_r\rho L} d\rho = \frac{-Q}{2\pi\epsilon_0 L}\int_b^a \frac{d\rho}{\rho\frac{(10+\rho)}{\rho}} = \frac{-Q}{2\pi\epsilon_0 L}\int_b^a \frac{d\rho}{10+\rho} \quad \left[\because V = -\int E \cdot dl\right]$$

$[L = 1]$

$$V = \frac{-Q}{2\pi\epsilon_0 L} \ln(10+\rho)\Big|_b^a = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{10+b}{10+a}\right) = \frac{Q}{2\pi\epsilon_0}\ln\left(\frac{12.5}{11}\right) = \frac{Q}{2\pi\epsilon_0}\ln(0.127)$$

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0}{\ln(0.127)} = \frac{2\pi \cdot 10^{-9}}{36\pi \ln(0.127)} = 434.6\,pF \quad \boxed{\therefore C = 434.6\,pF}$$

Problem

A spherical capacitor with $a = 1.5cm$ & $b = 4cm$ has an inhomogeneous dielectric of $\epsilon = \frac{10\epsilon_0}{r}$. calculate the capacitance of capacitor $\boxed{\text{Ans: } 1.13\,nF}$

Problem

Given the electric flux density $D = 0.3r\,a_r\,nc/m^2$ in free space (i) Find E at $P(r=2, \theta=25°, \phi=90°)$. (ii) Find the total charge with in the sphere of radius $r = 3$ (iii) Find the total electric flux leaving the sphere of radius $r = 4$.

Sol

(i) $D = \epsilon_0 E \Rightarrow E = \frac{D}{\epsilon_0} = \frac{(0.3r)a_r\,nc/m^2}{\epsilon_0} = \frac{0.3(2)\,a_r \times 10^{-9}}{10^{-9}/36\pi} = 0.6 \times 36\pi = 67.85\ V/m$

(ii) $Q = \oint D \cdot ds = \oint (0.3r)a_r \cdot r^2 \sin\theta\, d\theta\, d\phi\, a_r = \int_{\phi=0}^{2\pi} d\phi \cdot \int_{\theta=0}^{\pi} \sin\theta\, d\theta \cdot (0.3) r^3 (1)$

$Q = [\phi]_0^{2\pi} [-\cos\theta]_0^{\pi} (0.3)(3)^3 = 2\pi \cdot 2 \cdot (0.3) \cdot (27) = 101.78\,nC$

(iii) $Q = \Psi\ (r=4)$:

$Q = D(4\pi r^2) = (0.3)\,r \cdot a_r\,(4\pi r^2) = (0.3)\,4\pi r^3$

$Q = (0.3)\,4\pi\,(4)^3 = 241.27\,nc$

$\frac{Q}{4\pi r^2} = D$

$D = D_r a_r$

$D = D_r \cdot 1$

Problem

Find the total charge inside each of the volume indicated

(i) $\rho_v = 10z^2 e^{-0.1x}(\sin\pi y)$, $-1 \le x \le 2$, $0 \le y \le 1$, $3 \le z \le 3.6$

(ii) $\rho_v = 4xyz^2$, $0 \le \rho \le 2$, $0 \le \phi \le \pi/2$, $0 \le z \le 3$

(iii) $\rho_v = \frac{3\pi \cos^2\theta \cos^2\phi}{2r^2(r^2+1)}$